MATH 205 FINAL EXAM APRIL 10, 2007

NAME: ______________________________

YOUR GRADE IS BASED ON CORRECTNESS, COMPLETENESS, AND CLARITY ON EACH EXERCISE. YOU MAY USE A CALCULATOR AND THE AGREED-UPON PAGE OF NOTES, BUT NO BOOKS OR OTHER STUDENTS. GOOD LUCK!

1.) (10 pts.) Given the quadratic form $8x_1^2 + 6x_1x_2$,

a.) (2 pts.) find the symmetric matrix of the quadratic form;

b.) (2 pts.) classify the quadratic form as positive definite, negative definite, or indefinite, and explain your reasoning;

c.) (6 pts.) make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form into one with no cross-product term.

2.) (15 pts.)

a.) (5 pts.) **True or False:** A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a vector $\hat{\mathbf{x}}$ such that $\|\mathbf{b} - A\mathbf{x}\| \leq \|\mathbf{b} - A\hat{\mathbf{x}}\|$ for all $\mathbf{x}$ in $\mathbb{R}^n$. If this is true, explain why. If it is false, correct the statement to make it true.

b.) (10 pts.) While boiling a pot of water, you take the temperature every two minutes. This generates the data points $(0, 15)$, $(2, 37)$, $(4, 68)$, $(6, 89)$, where the first coordinate is time, in minutes, and the second coordinate is temperature, in degrees Celsius. Using linear algebra techniques, find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits these data points.

3.) (15 pts.)

a.) (5 pts.) Is it true that $\mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} = 0$ for every pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$? If so, explain why; if not, explain why not.

b.) (5 pts.) Suppose both U and V are orthogonal matrices. Explain why UV is an orthogonal matrix. That is, explain why UV is invertible and why its inverse is $(UV)^T$.

c.) (5 pts.) The Orthogonal Decomposition Theorem gives a formula for computing $\hat{\mathbf{y}}$, the projection of a vector $\mathbf{y}$ onto a subspace W of $\mathbb{R}^n$. Each term in that formula is itself a projection of $\mathbf{y}$ onto a subspace of W. Explain why.

4.) (15 pts.)

a.) (5 pts.) Verify that $\mathbf{v} = (2, 1, -1, 2)$ is an eigenvector of A, given below. What is the corresponding eigenvalue of $\mathbf{v}$?

$$A = \begin{bmatrix} -6 & 4 & 0 & 9 \\ -3 & 0 & 1 & 6 \\ -1 & -2 & 1 & 0 \\ -4 & 4 & 0 & 7 \end{bmatrix}$$

b.) (5 pts.) Construct a 4×4 matrix with eigenvalues -3, 2, and 5 (with multiplicity 2). Your matrix should not be strictly diagonal - that is, there must be some nonzero entries in non-diagonal positions within the matrix.

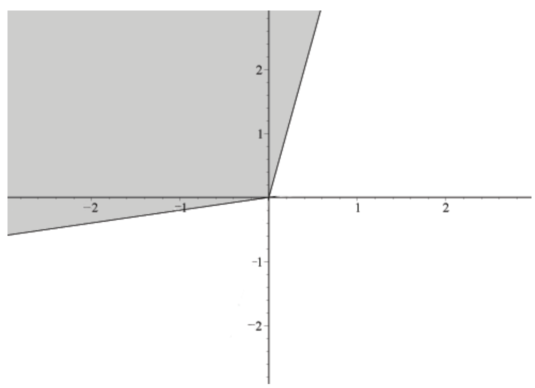
c.) (5 pts.) Use the factorization $A = PDP^{-1}$ to compute A^k, where k represents an arbitrary positive integer.

$$A = \begin{bmatrix} 33 & -20 \\ 60 & -37 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = PDP^{-1}$$

5.) (15 pts.)

a.) (5 pts.) What are the three properties of a subspace H of $\mathbb{R}^n$?

b.) (5 pts.) The shaded region in the image below is a set in $\mathbb{R}^2$. (Include the bounding lines as part of the set.) Give a specific reason why this set is *not* a subspace of $\mathbb{R}^2$.



c.) (5 pts.) Let A be an $m \times n$ matrix. Explain why Nul A is a subspace of $\mathbb{R}^m$.

6.) (15 pts.)

a.) (5 pts.) Must an elementary matrix be square? Why or why not?

b.) (5 pts.) Let T be the linear transformation $T(x_1, x_2, x_3) = (4x_2 - 6x_3,\ 0,\ 7x_2 - 9x_3,\ x_1)$. Find the matrix A for which $T(\mathbf{x}) = A\,\mathbf{x}$.

c.) (5 pts.) Use the matrix inverse algorithm to compute A^{-1}, if it exists. If it does not exist, explain how the algorithm shows this.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$$

7.) (15 pts.)

a.) (5 pts.) Suppose the vectors below are linearly independent. What can you say about the numbers a, b, c, d, e, and f?

$$\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

b.) (5 pts.) Write the coefficient matrix of the system of equations below.

$$\begin{array}{rcrcrcrcr} & & 3x_2 & - & 6x_3 & + & 8x_4 & = & -5 \\ 3x_1 & & & + & x_3 & - & 2x_4 & = & 7 \\ 4x_1 & + & x_2 & + & 5x_3 & & & = & 8 \end{array}$$

c.) (5 pts.) Write the augmented matrix of the system of equations in part (b). Does the system have a solution? How do you know?