NAME: Solutions

Math 205 - Final Exam - April 10, 2006

Instructions: Show enough work to justify your final answers.

1. (14 pts.) Let $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{u_1} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$, and let $W = \text{Span}\{\mathbf{u_1}, \mathbf{u_2}\}$.

(a) Is $\mathbf{y}$ in W? Are there constants c_1, c_2 such that

$c_1\vec{u}_1 + c_2\vec{u}_2 = \vec{y}$?

ie $\begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

Augmented matrix: $\begin{bmatrix} 1 & 5 & 1 \\ 3 & 1 & 3 \\ -2 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 1 \\ 0 & -14 & 0 \\ 0 & 14 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 1 \\ 0 & -14 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

Pivot in last column, so there is no solution.

Thus, $\vec{y}$ is not in $\text{Span}\{\vec{u}_1, \vec{u}_2\}$.

(b) Find the vector in W that is closest to $\mathbf{y}$.

Need $\text{proj}_W\vec{y} = \hat{y}$. Since $\vec{u}_1 \cdot \vec{u}_2 = 5+3-8=0$, have orthogonal basis for W, so $\text{proj}_W\vec{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 = \frac{0}{14}\vec{u}_1 + \frac{28}{42}\vec{u}_2$

$= \frac{2}{3}\vec{u}_2 = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix}$.

(c) Find a vector in $W^{\perp}$.

For $\vec{y} = \hat{y} + \vec{z}$, $\vec{z}$ is in $W^{\perp}$.

$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} = \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix}$. (Any scalar multiple of this vector is also in $W^{\perp}$, so, for instance, $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is in $W^{\perp}$, too.)

2. (14 pts.) Consider the matrix $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$.

(a) Show that A is diagonalizable. That is, find matrices P and D and write the equation involving A, P, and D.

$$|A - \lambda I| = \begin{vmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{vmatrix} = (7-\lambda)(1-\lambda) - 2(-4)$$
$$= \lambda^2 - 8\lambda + 7 + 8 = \lambda^2 - 8\lambda + 15$$
$$= (\lambda - 5)(\lambda - 3)$$

Set $(\lambda-5)(\lambda-3) = 0$. $\rightarrow$ $\lambda = 5$ or $\lambda = 3$ are the eigenvalues.

$\underline{\lambda = 5}$: $(A - 5I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 2 & 2 & 0 \\ -4 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad \begin{matrix} x_1 = -x_2 \\ x_2 \text{ free.} \end{matrix} \quad \vec{x} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

An Eigenvector for $\lambda = 5$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$\underline{\lambda = 3}$: $(A - 3I)\vec{x} = \vec{0}$. $\begin{bmatrix} 4 & 2 & 0 \\ -4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $\begin{matrix} x_1 = -\frac{1}{2}x_2 \\ x_2 \text{ free} \end{matrix}$. $\vec{x} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

An eigenvector for $\lambda = 3$ is $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$.

$\rightarrow P = \begin{bmatrix} -1 & -1/2 \\ 1 & 1 \end{bmatrix}$. $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$. $A = PDP^{-1}$.

(b) Using your result in part (a), simplify A^k. (Your answer should be a single matrix.)

$A^k = PD^kP^{-1}$. $\quad P^{-1} = \frac{1}{|P|} \begin{bmatrix} 1 & 1/2 \\ -1 & -1 \end{bmatrix}$. $\quad |P| = -1 + \frac{1}{2} = -\frac{1}{2}$.

$$= -2 \begin{bmatrix} 1 & 1/2 \\ -1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix}.$$

So
$$A^k = \begin{bmatrix} -1 & -1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} -5^k & -\frac{1}{2}3^k \\ 5^k & 3^k \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2\cdot 5^k - 3^k & 5^k - 3^k \\ -2\cdot 5^k + 2\cdot 3^k & -5^k + 2\cdot 3^k \end{bmatrix}.$$

3. (14 pts.) Consider the data points $(-1, 0)$, $(0, 3)$, $(1, 4)$, and $(2, 3)$. Find the values of a and b so that the equation $y = ax + b$ of the least-squares line best fits this data. (Hint: Write out the equations you would expect to be true if this line actually went through each data point.)

$y = ax + b.$

$(-1,0)$: $0 = a(-1)+b = -1a + 1b$

$(0,3)$: $3 = a(0)+b = 0a + 1b$

$(1,4)$: $4 = a(1)+b = 1a + 1b$

$(2,3)$: $3 = a(2)+b. = 2a + 1b.$

$$\rightarrow \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 3 \\ 4 \\ 3 \end{bmatrix}}_{\vec{b}}.$$

$A \quad \vec{x} = \vec{b}.$

Least-squares solution?
Normal equations: $A^T A\vec{x} = A^T\vec{b}.$

$$A^T A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}.$$

$$A^T\vec{b} = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

$$\rightarrow \begin{bmatrix} 6 & 2 & 10 \\ 2 & 4 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -5 & -10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$ So $a = 1$, $b = 2$.

Line : $\boxed{y = 1x + 2}$ best fits the data.

4. (12 pts.) Suppose A is a 4×4 matrix with eigenvalues -3, 0, and 2. Assume that the eigenspace of $\lambda = 2$ is 2-dimensional.

(a) Is A invertible? Why or why not?

$\lambda = 0$ is an eigenvalue, so $|A - 0I| = 0$, so $|A| = 0$,

so A is not invertible (by Invertible Matrix Thm.)

(b) Is A diagonalizable? Why or why not?

Eigenvectors from different eigenspaces are independent. Since there are 2 independent eigenvectors for $\lambda = 2$, there are a total of 4 ~~Ei~~ independent eigenvectors from $\lambda = -3$, $\lambda = 0$, $\lambda = 2$, so A is diagonalizable.

(c) Suppose **u** and **v** are in the eigenspace of $\lambda = -3$. Is it possible that **u** and **v** are linearly independent?

Eigenspace of $\lambda = -3$ is 1-dimensional, so $\vec{u}$ and $\vec{v}$ are necessarily scalar multiples of each other. Thus, they cannot be linearly independent.

(d) Suppose **w** is in the eigenspace of $\lambda = 2$. Calculate $A^5\mathbf{w}$.

$A\vec{w} = 2\vec{w}$. So $A^5\vec{w} = 2^5\vec{w} = 32\vec{w}$.

5. (6 pts.) Suppose that λ is a non-zero eigenvalue of an invertible matrix A. Show that $1/\lambda$ is an eigenvalue of A^{-1}. (Hint: Consider the equation $A\mathbf{x} = \lambda\mathbf{x}$.)

$$A\vec{x} = \lambda\vec{x}.$$

$$A^{-1}A\vec{x} = A^{-1}(\lambda\vec{x})$$

$$\vec{x} = \lambda A^{-1}\vec{x}.$$

$$\lambda^{-1}\vec{x} = A^{-1}\vec{x}.$$ So λ^{-1} is an eigenvalue of A^{-1}.

6. (14 pts.) Consider the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$, where $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

The eigenvalues are 5, 2, and -1, with corresponding eigenvectors $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

(a) Write out $Q(\mathbf{x})$ in terms of x_1, x_2, and x_3.

$Q(\vec{x}) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$.

(b) Find a matrix P such that the change of variables $\mathbf{x} = P\mathbf{y}$ transforms the quadratic form into one with no cross-product term.

The eigenvectors $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ are orthogonal.

They need to be orthonormal to form the columns of P.

$\left\| \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right\| = \sqrt{9} = 3.$ So divide each entry by 3.

$\rightarrow P = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}.$

(c) Write the new quadratic form with no cross-product term.

$Q(\vec{x}) = Q(P\vec{y}) = (P\vec{y})^T A P\vec{y} = \vec{y}^T P^T A P \vec{y} = \vec{y}^T D \vec{y}$.

$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. $Q(\vec{x}) = 5y_1^2 + 2y_2^2 - 1y_3^2$.

(d) Bonus: Find a vector $\mathbf{x}$ such that $Q(\mathbf{x})$ is negative.

For $\vec{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $Q(\vec{x}) = 0 + 0 - 1 = -1$.

Corresponding $\vec{x} = P\vec{y} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$. (Note: many other vectors work..)

7. (10 pts.) Miscellaneous problems.

(a) Suppose A is 4×4 and $\det(A) = 3$. What is $\det(2A)$? (Careful.)

$|2A| = 2^4 |A| = 16 \cdot 3 = 48.$

(b) If 3 is an eigenvalue of C, what is $\det(C - 3I)$?

If λ is an eigenvalue of C, then $|C - \lambda I| = 0$.

Hence, $|C - 3I| = 0$.

(c) Let $B = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$.

i. Are the columns of B orthogonal?

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} = 6 - 6 = 0.$

Yes.

ii. Is B an orthogonal matrix?

No, an orthogonal matrix has orthonormal columns.

iii. Is B orthogonally diagonalizable?

B is symmetric, so it is orthogonally diagonalizable.

8. (8 pts.) Consider the vectors $\mathbf{u_1} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{u_2} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Let $W = \text{Span}\{\mathbf{u_1}, \mathbf{u_2}\}$. Find an orthogonal basis for W.

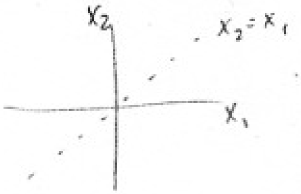
Let $\vec{v}_1 = \vec{u}_1$.

$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$

So ~~the~~ an orthogonal basis is $\{\vec{v}_1, \vec{v}_2\} = \left\{\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right\}$.

9. (12 pts.) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(\mathbf{x})$ reflects points about the line $x_2 = x_1$. Let A be the standard matrix of the transformation T.

(a) Find two linearly independent eigenvectors of A, and give their corresponding eigenvalues.



Any vector on the line $x_2 = x_1$ is mapped to itself.

Any vector on the line $x_2 = -x_1$ is mapped to the negative of itself.

Thus, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 1.

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue -1.

(Calculations can be done for $A = [T(\vec{e}_1)\ T(\vec{e}_2)] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.)

(b) Is T a one-to-one transformation? Explain.

Since $T(\vec{x}) = A\vec{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{x}$, and since A is invertible,

by the Invertible Matrix Theorem, T is one-to-one.

(c) Is T an onto transformation? Explain.

As in b), IMT also says that T is onto.

1	2	3	4	5	6	7	8	9	TOTAL