

Math 205 - Final Exam - April 10, 2006

Instructions: Show enough work to justify your final answers.

1. (14 pts.) Let $\vec{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$, and let $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$.

(a) Is \vec{y} in W ? Are there constants c_1, c_2 such that

$$c_1 \vec{u}_1 + c_2 \vec{u}_2 = \vec{y}?$$

$$\text{ie } \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

$$\text{Augmented matrix: } \begin{bmatrix} 1 & 5 & 1 \\ 3 & 1 & 3 \\ -2 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 1 \\ 0 & -14 & 0 \\ 0 & 14 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 1 \\ 0 & -14 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

Pivot in last column, so there is no solution.

Thus, \vec{y} is not in $\text{Span}\{\vec{u}_1, \vec{u}_2\}$.

(b) Find the vector in W that is closest to \vec{y} .

Need $\text{proj}_W \vec{y} = \hat{\vec{y}}$. Since $\vec{u}_1 \cdot \vec{u}_2 = 5 + 3 - 8 = 0$, have orthogonal basis for W , so

$$\text{proj}_W \vec{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 = \frac{0}{14} \vec{u}_1 + \frac{28}{42} \vec{u}_2 = \frac{2}{3} \vec{u}_2 = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix}$$

(c) Find a vector in W^\perp .

For $\vec{y} = \hat{\vec{y}} + \vec{z}$, \vec{z} is in W^\perp .

$$\vec{z} = \vec{y} - \hat{\vec{y}} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} = \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix}. \text{ (Any scalar multiple of this vector is also in } W^\perp, \text{ so, for instance, } \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \text{ is in } W^\perp, \text{ too.)}$$

2. (14 pts.) Consider the matrix $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$.

(a) Show that A is diagonalizable. That is, find matrices P and D and write the equation involving A , P , and D .

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{vmatrix} = (7-\lambda)(1-\lambda) - 2(-4) \\ &= \lambda^2 - 8\lambda + 7 + 8 = \lambda^2 - 8\lambda + 15 \\ &= (\lambda - 5)(\lambda - 3) \end{aligned}$$

Set $(\lambda - 5)(\lambda - 3) = 0$. $\rightarrow \lambda = 5$ or $\lambda = 3$ are the eigenvalues.

$\lambda = 5$: $(A - 5I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 2 & 2 & 0 \\ -4 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = -x_2 \\ x_2 \text{ free.} \end{array} \quad \vec{x} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

An eigenvector for $\lambda = 5$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$\lambda = 3$: $(A - 3I)\vec{x} = \vec{0}$. $\begin{bmatrix} 4 & 2 & 0 \\ -4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $\begin{array}{l} x_1 = -\frac{1}{2}x_2 \\ x_2 \text{ free.} \end{array} \quad \vec{x} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

An eigenvector for $\lambda = 3$ is $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$.

$$\rightarrow P = \begin{bmatrix} -1 & -1/2 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \quad A = PDP^{-1}$$

(b) Using your result in part (a), simplify A^k . (Your answer should be a single matrix.)

$$\begin{aligned} A^k &= P D^k P^{-1} \quad P^{-1} = \frac{1}{|P|} \begin{bmatrix} 1 & 1/2 \\ -1 & -1 \end{bmatrix} \quad |P| = -1 + \frac{1}{2} = -\frac{1}{2} \\ &= -2 \begin{bmatrix} 1 & 1/2 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{So } A^k &= \begin{bmatrix} -1 & -1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -5^k & -\frac{1}{2}3^k \\ 5^k & 3^k \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 5^k - 3^k & 5^k - 3^k \\ -2 \cdot 5^k + 2 \cdot 3^k & -5^k - 2 \cdot 3^k \end{bmatrix} \end{aligned}$$

3. (14 pts.) Consider the data points $(-1, 0)$, $(0, 3)$, $(1, 4)$, and $(2, 3)$. Find the values of a and b so that the equation $y = ax + b$ of the least-squares line best fits this data. (Hint: Write out the equations you would expect to be true if this line actually went through each data point.)

$$y = ax + b.$$

$$(-1, 0) : 0 = a(-1) + b = -1a + 1b$$

$$(0, 3) : 3 = a(0) + b = 0a + 1b$$

$$(1, 4) : 4 = a(1) + b = 1a + 1b$$

$$(2, 3) : 3 = a(2) + b = 2a + 1b$$

$$\rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 4 \\ 3 \end{bmatrix}$$

$$A \vec{x} = \vec{b}$$

Least-squares solution?

Normal equations:

$$A^T A \vec{x} = A^T \vec{b}$$

$$A^T A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 6 & 2 & 10 \\ 2 & 4 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -5 & -10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

So $a = 1$, $b = 2$.

Line: $y = 1x + 2$ best fits the data.

4. (12 pts.) Suppose A is a 4×4 matrix with eigenvalues -3 , 0 , and 2 . Assume that the eigenspace of $\lambda = 2$ is 2-dimensional.

(a) Is A invertible? Why or why not?

$\lambda = 0$ is an eigenvalue, so $|A - 0I| = 0$, so $|A| = 0$,
so A is not invertible (by Invertible Matrix Thm.)

(b) Is A diagonalizable? Why or why not?

Eigenvectors from different eigenspaces are independent. Since there are 2 independent eigenvectors for $\lambda = 2$, there are a total of 4 independent eigenvectors from $\lambda = -3$, $\lambda = 0$, $\lambda = 2$, so A is diagonalizable.

(c) Suppose u and v are in the eigenspace of $\lambda = -3$. Is it possible that u and v are linearly independent?

Eigenspace of $\lambda = -3$ is 1-dimensional, so \vec{u} and \vec{v} are necessarily scalar multiples of each other. Thus, they cannot be linearly independent.

(d) Suppose w is in the eigenspace of $\lambda = 2$. Calculate $A^5 w$.

$$A\vec{w} = 2\vec{w}. \quad \text{So } A^5\vec{w} = 2^5\vec{w} = 32\vec{w}.$$

5. (6 pts.) Suppose that λ is a non-zero eigenvalue of an invertible matrix A . Show that $1/\lambda$ is an eigenvalue of A^{-1} . (Hint: Consider the equation $Ax = \lambda x$.)

$$A\vec{x} = \lambda\vec{x}.$$

$$A^{-1}A\vec{x} = A^{-1}(\lambda\vec{x})$$

$$\vec{x} = \lambda A^{-1}\vec{x}.$$

$$\lambda^{-1}\vec{x} = A^{-1}\vec{x}. \quad \text{So } \lambda^{-1} \text{ is an eigenvalue of } A^{-1}.$$

6. (14 pts.) Consider the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$, where $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

The eigenvalues are 5, 2, and -1, with corresponding eigenvectors $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

(a) Write out $Q(\mathbf{x})$ in terms of x_1 , x_2 , and x_3 .

$$Q(\mathbf{x}) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3.$$

(b) Find a matrix P such that the change of variables $\mathbf{x} = P\mathbf{y}$ transforms the quadratic form into one with no cross-product term.

The eigenvectors $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ are orthogonal.

They need to be orthonormal to form the columns of P .

$$\left\| \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right\| = \sqrt{9} = 3. \quad \text{So divide each entry by 3.}$$

$$\rightarrow P = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}.$$

(c) Write the new quadratic form with no cross-product term.

$$Q(\mathbf{x}) = Q(P\mathbf{y}) = (P\mathbf{y})^T A P\mathbf{y} = \mathbf{y}^T P^T A P \mathbf{y} = \mathbf{y}^T D \mathbf{y}.$$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad Q(\mathbf{x}) = 5y_1^2 + 2y_2^2 - 1y_3^2.$$

(d) Bonus: Find a vector \mathbf{x} such that $Q(\mathbf{x})$ is negative.

$$\text{For } \mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad Q(\mathbf{x}) = 0 + 0 - 1 = -1.$$

$$\text{Corresponding } \mathbf{x} = P\mathbf{y} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}. \quad (\text{Note: many other vectors work...})$$

7. (10 pts.) Miscellaneous problems.

(a) Suppose A is 4×4 and $\det(A) = 3$. What is $\det(2A)$? (Careful.)

$$|2A| = 2^4 |A| = 16 \cdot 3 = 48.$$

(b) If 3 is an eigenvalue of C , what is $\det(C - 3I)$?

If λ is an eigenvalue of C , then $|C - \lambda I| = 0$.

Hence, $|C - 3I| = 0$.

(c) Let $B = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$.

i. Are the columns of B orthogonal?

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} = 6 - 6 = 0.$$

Yes.

ii. Is B an orthogonal matrix?

No, an orthogonal matrix has orthonormal columns.

iii. Is B orthogonally diagonalizable?

B is symmetric, so it is orthogonally diagonalizable.

8. (8 pts.) Consider the vectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Let $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Find an orthogonal basis for W .

Let $\vec{v}_1 = \vec{u}_1$.

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

So ^{an} orthogonal basis is $\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$.

