NAME: ____

Math 205 - Final Exam - April 10, 2006

Instructions: Show enough work to justify your final answers.

1. (14 pts.) Let $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{u_1} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$, and let $W = \text{Span}\{\mathbf{u_1}, \mathbf{u_2}\}$.

   (a) Is $\mathbf{y}$ in W?

   (b) Find the vector in W that is closest to $\mathbf{y}$.

   (c) Find a vector in $W^{\perp}$.

2. (14 pts.) Consider the matrix $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$.

(a) Show that A is diagonalizable. That is, find matrices P and D and write the equation involving A, P, and D.

(b) Using your result in part (a), simplify A^k. (Your answer should be a single matrix.)

3. (14 pts.) Consider the data points $(-1, 0)$, $(0, 3)$, $(1, 4)$, and $(2, 3)$. Find the values of a and b so that the equation $y = ax + b$ of the least-squares line best fits this data. (Hint: Write out the equations you would expect to be true if this line actually went through each data point.)

4. (12 pts.) Suppose A is a 4×4 matrix with eigenvalues -3, 0, and 2. Assume that the eigenspace of $\lambda = 2$ is 2-dimensional.

   (a) Is A invertible? Why or why not?

   (b) Is A diagonalizable? Why or why not?

   (c) Suppose $\mathbf{u}$ and $\mathbf{v}$ are in the eigenspace of $\lambda = -3$. Is it possible that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent?

   (d) Suppose $\mathbf{w}$ is in the eigenspace of $\lambda = 2$. Calculate $A^5\mathbf{w}$.

5. (6 pts.) Suppose that λ is a non-zero eigenvalue of an invertible matrix A. Show that $1/\lambda$ is an eigenvalue of A^{-1}. (Hint: Consider the equation $A\mathbf{x} = \lambda\mathbf{x}$.)

6. (14 pts.) Consider the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, where $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

The eigenvalues are 5, 2, and -1, with corresponding eigenvectors $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

(a) Write out $Q(\mathbf{x})$ in terms of x_1, x_2, and x_3.

(b) Find a matrix P such that the change of variables $\mathbf{x} = P\mathbf{y}$ transforms the quadratic form into one with no cross-product term.

(c) Write the new quadratic form with no cross-product term.

(d) Bonus: Find a vector $\mathbf{x}$ such that $Q(\mathbf{x})$ is negative.

7. (10 pts.) Miscellaneous problems.

(a) Suppose A is 4×4 and $\det(A) = 3$. What is $\det(2A)$? (Careful.)

(b) If 3 is an eigenvalue of C, what is $\det(C - 3I)$?

(c) Let $B = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$.

i. Are the columns of B orthogonal?

ii. Is B an orthogonal matrix?

iii. Is B orthogonally diagonalizable?

8. (8 pts.) Consider the vectors $\mathbf{u_1} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{u_2} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Let W =Span$\{\mathbf{u_1}, \mathbf{u_2}\}$. Find an orthogonal basis for W.

9. (12 pts.) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(\mathbf{x})$ reflects points about the line $x_2 = x_1$. Let A be the standard matrix of the transformation T.

(a) Find two linearly independent eigenvectors of A, and give their corresponding eigenvalues.

(b) Is T a one-to-one transformation? Explain.

(c) Is T an onto transformation? Explain.

1	2	3	4	5	6	7	8	9	TOTAL