1. (12 points) Solve the IVP $y' = \dfrac{x \sin x}{3y^2}$, $y(\pi) = 0$.

$$\frac{dy}{dx} = \frac{x\sin x}{3y^2}, \text{ so } 3y^2\,dy = x\sin x\,dx, \text{ so } \int 3y^2\,dy = \int x\sin x\,dx.$$

$$\int 3y^2\,dy = y^3 + C_1. \qquad \int x\sin x\,dx = -x\cos x - \int -\cos x\,dx$$

$u = x \quad dv = \sin x\,dx$

$du = dx \quad v = -\cos x$

$$= -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C_2.$$

So, $y^3 + C_1 = -x\cos x + \sin x + C_2$

So $y^3 = -x\cos x + \sin x + C$ (where $C = C_2 - C_1$)

So $y = \sqrt[3]{-x\cos x + \sin x + C}$. Then, $y(\pi) = 0$? when $x = \pi$, $y = 0$.

$0 = \sqrt[3]{-\pi\cos(\pi) + \sin(\pi) + C}$

$0 = \sqrt[3]{\pi + 0 + C} \longrightarrow C = -\pi$.

Thus, $\boxed{y = \sqrt[3]{-x\cos x + \sin x - \pi}}$

2. (12 points) Evaluate $\displaystyle\int \frac{dx}{4 - x^2}$.

Method 1: $\displaystyle\int \frac{dx}{4-x^2} = \int \frac{dx}{(2+x)(2-x)}$

$$\frac{1}{(2+x)(2-x)} = \frac{A}{2+x} + \frac{B}{2-x}$$

$1 = A(2-x) + B(2+x)$

$x=2$: $1 = A(0) + B(4)$.

$\rightarrow B = 1/4$.

$x=-2$: $1 = A(4) + B(0)$

$\rightarrow A = 1/4$.

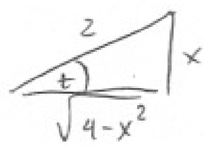
Thus, $\displaystyle\int \frac{dx}{4-x^2} = \int \frac{1/4}{2+x} + \frac{1/4}{2-x}\,dx$

$\displaystyle = \frac{1}{4}\int \frac{1}{2+x} + \frac{1}{2-x}\,dx$

$\boxed{= \frac{1}{4}\left(\ln|2+x| - \ln|2-x|\right) + C}$

Method 2: $\displaystyle\int \frac{dx}{4-x^2}$. Let $x = 2\sin t$, $dx = 2\cos t\,dt$.

$\sin t = \frac{x}{2}$.



$\displaystyle = \int \frac{2\cos t}{4 - 4\sin^2 t}$

$\displaystyle = \frac{2}{4}\int \frac{\cos t\,dt}{1-\sin^2 t}$

$\displaystyle = \frac{1}{2}\int \frac{\cos t\,dt}{\cos^2 t}$

$\displaystyle = \frac{1}{2}\int \frac{dt}{\cos t}$

$\displaystyle = \frac{1}{2}\int \sec t\,dt$

$= \frac{1}{2}\ln|\sec t + \tan t| + C$

$\boxed{= \frac{1}{2}\ln\left|\frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}}\right| + C}$

(Yes, these are equal!)

3. (8 points) Write down an integral that gives the length of the curve $y = x^2$ from $x = 0$ to $x = 2$. (Do not evaluate the integral.)

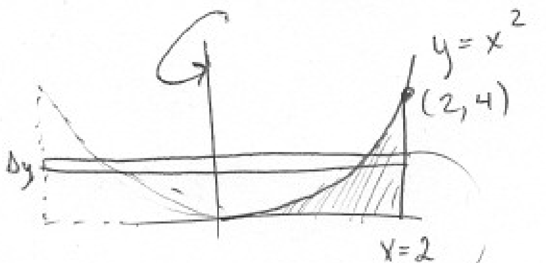
Arclength of $y=f(x)$ from $x=a$ to $x=b$ is given by

$$\int_a^b \sqrt{1+f'(x)^2}\,dx.$$

Here, $f(x)=x^2$, so $f'(x)=2x$, so $f'(x)^2 = 4x^2$, so

$$\boxed{\text{arclength} = \int_0^2 \sqrt{1+4x^2}\,dx.}$$

4. (12 points) Consider the region in the xy-plane that is bounded by the graphs of $y = x^2$, $y = 0$, and $x = 2$. Suppose this region is rotated around the y-axis. Write down an integral that is equal to the volume of the resulting solid. (Do not evaluate the integral.)



Slice.

$$A_{slice} = \pi R^2 - \pi r^2$$
$$= \pi \cdot 2^2 - \pi \cdot x^2$$
$$= 4\pi - \pi x^2$$

$$V_{slice} = (4\pi - \pi x^2)\Delta y.$$
$$= (4\pi - \pi y)\Delta y$$
since $y = x^2$.

$$\boxed{V_{total} = \int_{y=0}^{y=4} (4\pi - \pi y)\,dy.}$$

5. (10 points) Find the radius and interval of convergence of $\sum_{k=0}^{\infty} \frac{(x-1)^k}{(k+1)4^k}$.

Ratio test:

$$L = \lim_{k\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{k\to\infty} \left| \frac{(x-1)^{k+1}}{(k+1+1)4^{k+1}} \cdot \frac{(k+1)4^k}{(x-1)^k} \right| = \lim_{k\to\infty} \left| \frac{(x-1)(k+1)}{(k+2)\cdot 4} \right|$$

$$= \left|\frac{x-1}{4}\right| \lim_{k\to\infty} \left|\frac{k+1}{k+2}\right| \cdot \frac{1/k}{1/k} = \left|\frac{x-1}{4}\right| \lim_{k\to\infty} \left|\frac{1+1/k}{1+2/k}\right| = \left|\frac{x-1}{4}\right| \cdot \left|\frac{1+0}{1+0}\right|$$

$$= \left|\frac{x-1}{4}\right| = \frac{|x-1|}{4}.$$ Need $L<1$, so $\frac{|x-1|}{4} < 1$, $|x-1|<4$.

Power series converges for $|x-1|<4$, diverges for $|x-1|>4$, and we need to check endpts for $|x-1|=4$. (Note: Radius of conv $=4$.)

$|x-1|<4 \rightarrow -3<x<5$. Check $x=-3$, $x=5$.

$\underline{x=-3}$: $$\sum_{k=0}^{\infty}\frac{(x-1)^k}{(k+1)4^k} = \sum_{k=0}^{\infty}\frac{(-3-1)^k}{(k+1)4^k} = \sum_{k=0}^{\infty}\frac{(-4)^k}{(k+1)4^k} = \sum_{k=0}^{\infty}\frac{(-1)^k 4^k}{(k+1)4^k} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}$$

$= \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, alternating harmonic series, which $\underline{\text{converges}}$.

$\underline{x=5}$: $$\sum_{k=0}^{\infty}\frac{(5-1)^k}{(k+1)4^k} = \sum_{k=0}^{\infty}\frac{4^k}{(k+1)4^k} = \sum_{k=0}^{\infty}\frac{1}{k+1} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots,$$ harmonic series, which $\underline{\text{diverges}}$.

Hence, include $x=-3$, exclude $x=5$.

$$\boxed{\begin{array}{l}\text{Interval of convergence} = [-3,5).\\ \text{Radius of convergence} = 4.\end{array}}$$

6. (8 points) Evaluate the following series or explain why it diverges: $\sum_{k=1}^{\infty} \frac{1}{3^{k+1}}$. State the name of the convergence test that you use. (If you determine that this series converges, give the exact value that it converges to.)

$$\sum_{k=1}^{\infty} \frac{1}{3^{(k+1)}} = \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots$$, geometric series with $r = \frac{1}{3}$.

Since $|r| < 1$, this converges to $\frac{a}{1-r}$, where a is the first term in the series $(= \frac{1}{3^2})$.

So $$\sum_{k=1}^{\infty} \frac{1}{3^{k+1}} = \frac{a}{1-r} = \frac{1/9}{1-1/3} = \frac{1/9}{2/3} = \frac{1}{9} \cdot \frac{3}{2} = \boxed{\frac{1}{6}}$$

7. (8 points each) Determine whether the following series converge or diverge, and state the name of the convergence test that you use.

(a) $\sum_{k=0}^{\infty} \frac{k+3}{2k+1}$.

$$a_k = \frac{k+3}{2k+1}. \quad \lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{k+3}{2k+1} = \lim_{k \to \infty} \frac{1 + 3/k}{2 + 1/k} = \frac{1+0}{2+0} = 1/2.$$

By the "nth term test", since $\lim_{k \to \infty} a_k \neq 0$, the series diverges.

(b) $\sum_{k=1}^{\infty} \frac{\arctan k}{1+k^2}$.

Integral test:

$u = \arctan x$
$du = \frac{1}{1+x^2} dx$

$f(x) = \frac{\arctan x}{1+x^2}$.

$$\int_1^{\infty} \frac{\arctan x}{1+x^2} dx = \lim_{t \to \infty} \int_1^t \frac{\arctan x}{1+x^2} dx$$

$$= \lim_{t \to \infty} \int_{x=1}^{x=t} u \, du = \lim_{t \to \infty} \frac{u^2}{2} \Big]_{x=1}^{x=t}$$

$$= \lim_{t \to \infty} \frac{(\arctan x)^2}{2} \Big]_{x=1}^{x=t} = \lim_{t \to \infty} \frac{(\arctan t)^2}{2} - \frac{(\arctan 1)^2}{2}$$

$$= \frac{(\pi/2)^2}{2} - \frac{(\pi/4)^2}{2} = \frac{3\pi^2}{32},$$ finite, so converges, so the series converges, too.

8. (10 points) Find the power series centered at $x_0 = 0$ for $f(x) = \ln(1+x)$. You do not need to calculate the radius or interval of convergence.

(Hint: There are two ways to do this - one by calculating the Taylor series for $f(x)$, and the other by starting with the power series of $\frac{1}{1-x}$ and manipulating it.)

Method 1: Know $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$, so ~~#~~

So $\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots$

So $\ln(1+x) = \int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + \cdots)\,dx = (x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots) + C.$

When $x = 0$, $\ln(1+0) = \ln 1 = 0$. So, RHS is $0 - 0 + 0 - \cdots + C$. $\rightarrow C = 0$.

So $\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots}$

Method 2: (Taylor)

$f(x) = \ln(1+x) \rightarrow f(0) = 0$
$f'(x) = (1+x)^{-1} \rightarrow f'(0) = 1$
$f''(x) = -(1+x)^{-2} \rightarrow f''(0) = -1$
$f'''(x) = 2(1+x)^{-3} \rightarrow f'''(0) = 2$
$f''''(x) = -6(1+x)^{-4} \rightarrow f''''(0) = -6$

$\rightarrow$ Taylor series for $f(x) = \ln(1+x)$

$= 0 + \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{6}{4!}x^4 + \cdots$

$\boxed{= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots}$

9. (12 points) Let $g(x) = x^2 \sin(2x)$.

(a) Write out the first 4 terms of the power series of $g(x)$.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\sin(2x) = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots$

$= 2x - \frac{8x^3}{6} + \frac{2^5 x^5}{5!} - \frac{2^7 x^7}{7!} + \cdots$

So $\boxed{x^2 \sin(2x) = 2x^3 - \frac{8x^5}{6} + \frac{2^5 x^7}{5!} - \frac{2^7 x^9}{7!} + \cdots}$

(b) Evaluate $g^{(7)}(0)$.

$g^{(7)}(0) = 7! \cdot a_7 = \boxed{7! \cdot \frac{2^5}{5!}}$

coeff of x^7 from (a).

(c) Evaluate $g^{(8)}(0)$.

$g^{(8)}(0) = 8! \cdot a_8 = 8! \cdot 0 = \boxed{0}$

coeff of x^8, which is 0

Potentially useful formulæ

- $|I - L_n| \le \dfrac{K_1(b-a)^2}{2n}$.
- $|I - R_n| \le \dfrac{K_1(b-a)^2}{2n}$.
- $|I - T_n| \le \dfrac{K_2(b-a)^3}{12n^2}$.
- $|I - M_n| \le \dfrac{K_2(b-a)^3}{24n^2}$.

- $\displaystyle\int_a^b \sqrt{1+(f'(x))^2}\,dx$

- $\sin^2 x + \cos^2 x = 1$.
- $\tan^2 x + 1 = \sec^2 x$.
- $\frac{d}{dx}(\tan x) = \sec^2 x$.
- $\frac{d}{dx}(\sec x) = \sec x \tan x$.

- $\displaystyle\int \sin^n(x)\,dx = \frac{-\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$, for $n > 0$.
- $\displaystyle\int \cos^n(x)\,dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$, for $n > 0$.
- $\displaystyle\int \tan^n(x)\,dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\,dx$, for $n \neq 1$.
- $\displaystyle\int \sec^n(x)\,dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$, for $n \neq 1$.
- $\displaystyle\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$.

- $\displaystyle P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$.
- $\displaystyle |f(x) - P_n(x)| \le \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}$.

- $\displaystyle \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ on $(-\infty, \infty)$.
- $\displaystyle \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ on $(-\infty, \infty)$.
- $\displaystyle \arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$ on $[-1, 1]$.