NAME____

I____II____III____IV____V____VI____VII____VIII____ IX____X____TOTAL _________

April 9, 2008	Mathematics 309 Abstract Algebra I Final Examination	Mr. Haines

(15) I. Define any three of these terms. Use a complete, mathematically correct sentence for each definition.

binary operation
group
ring
field
polynomial

A.

B.

C.

(10) II. Give examples of the following:

A. An integral domain with 71 elements.

B. An infinite cyclic subgroup of $\langle R^{*}, \cdot \rangle$, the non-zero real numbers under multiplication.

C. A field with 5 elements.

D. Two non-isomorphic abelian groups, each with 75 elements.

E. An integral domain that is not a field.

F. A polynomial of degree 71 in $\mathbb{Z}[x]$ that is irreducible in $\mathbb{Z}[x]$.

G. A non-trivial homomorphism from Z_9 to Z_6 .

H. A homomorphism from Z_4 to Z_8 whose kernel is $\{0\}$.

(5) III. Draw the lattice diagram of subgroups of Z_{27}, the integers modulo 27, where the operation is addition modulo 27.

(20) IV. Fill in the blanks:

A. A generator for $Z_8 \times Z_9$ different from (1, 1) is ______________________ .

B. The order of (1, 2, 3)(2, 3, 4, 5)(1, 2) in S_5 is ______________________ .

C. The number of zeroes of $x^3 + x + 1$ in $\mathbb{Z}_3$ is ________________.

D. The number of left cosets of $\langle 5 \rangle$ in Z_{20} is ________________.

E. The number of elements in the group S_7 is ____________________ .

F. The order of the group D_5 of symmetries of the regular pentagon is ___________ .

G. Express $(1, 5, 7, 3, 4, 2) \in S_7$ as a product of transpositions ________________ .

H. The order of $2 + \langle 4 \rangle$ in the group $Z_{12} / \langle 4 \rangle$ is ________________.

I. The index of A_5 in S_5 is ________________________________ .

J. The number of right cosets of $\langle 5 \rangle$ in Z_{10} is ________________.

K. The order of the group $(Z_4 \times Z_6) / \langle (2, 2) \rangle$ is ____________________ .

L. The units in the ring $\langle Z_9, +_9, \cdot_9 \rangle$ are ____________________ .

(5) V. Let $\phi : R \rightarrow R'$ be a ring homomorphism, which means that $\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\,\phi(b)$ for all $a, b \in R$.

Prove that if $a \in \ker(\phi)$ and $r \in R$, $ra \in \ker(\phi)$.

(5) VI. Factor $x^5 + x^3 + x + 1$ into irreducible factors over $\mathbb{Z}_2[x]$

(10) VII. Let G be a group and let H be a subgroup of G. Define the relation $\sim$ on G by rule $a \sim b$ if and only if $ab^{-1} \in H$. Prove that $\sim$ is an equivalence relation on G.

(10) VIII. The identity function, *id*, is one of four automorphisms of Z_8 . If the others are denoted ϕ_1, ϕ_2, and ϕ_3, complete this table:

x	$id(x)$	$\phi_1(x)$	$\phi_2(x)$	$\phi_3(x)$
0				
1				
2				
3				
4				
5				
6				
7				

The elements of this set of automorphisms of Z_8 (denoted *Aut* (Z_8)) a permutations of of the set {0, 1, 2, 3, 4, 5, 6, 7}. Therefore *Aut* (Z_8) is isomorphic to a group of permutations with 4 elements. Which of the two groups of order 4 is *Aut* (Z_8) ?

(10) IX. A Boolean ring B is a ring with the property that $a^2 = a$ for every $a \in$ B .

A. Prove that $a + a = 0$ for every $a \in$ B. [Hint: Expand $(a + a)^2$.]

B. Prove that $ab + ba = 0$ for all $a, b \in$ B . [Hint: Expand $(a + b)^2$.]

C. Prove that B is commutative. [Hint: Use Part B.]

(10) X. TRUE OR FALSE? (Don't guess! The number of incorrect responses will be subtracted from the number of correct ones. Thus, random guessing earns you no points at all.)

_____ 1. The real numbers are closed under subtraction.

_____ 2. The empty set is an example of a ring.

_____ 3. Every cyclic group has a generator.

_____ 4. The real numbers form an abelian group under addition.

_____ 5. The real numbers form a group under multiplication.

_____ 6. $Z_5 \times Z_{25}$ is a cyclic group.

_____ 7. The number of elements in any subgroup of a finite group G divides the number of elements in G .

_____ 8. Every permutation can be expressed as a product of cycles.

_____ 9. S_6 has no cyclic subgroups.

_____ 10. The composition of two permutations of a set A is always a permutation of A .

_____ 11. Every left coset of a subgroup of a group G is also a subgroup of G .

_____ 12. Every abelian group of order 8 contains a cyclic subgroup of order 8 .

_____ 13. Every finite group of prime order is cyclic

_____ 14. If $H \leq S_5$ and H has 24 elements,then S_5 / H is abelian.

_____ 15. The function $\phi : Z_6 \to Z_{12}$ defined by $\phi(n) = n$ for all $n \in Z_6$ is a homomorphism.

_____ 16. $x^5 + x^3 + x + 1$ is irreducible over $\mathbb{Z}_2[x]$.

_____ 17. Every subgroup of a group G is also a coset of G .

_____ 18. For any two groups G and G' there is a homomorphism from G to G' .

_____ 19. Every field is an integral domain.

_____ 20. Every cyclic group is finite.