MATH 205 FINAL EXAM April 9, 2008

Name: KEY

Your grade is based on correctness, completeness, and clarity on each exercise. You may use a calculator and your single sheet of paper with notes, but may not consult books or other students. Good luck!

1.) (10 pts.) Use theorems and concepts from Chapter 6 to address the following. Always explain your reasoning thoroughly.

a.) (5 pts.) Show that the only case in which a vector $\mathbf{x}$ can be in both the subspaces W and $W^{\perp}$ is when $\mathbf{x} = \mathbf{0}$.

If $\vec{x}$ is in W and $\vec{x}$ is in $W^{\perp}$ then $\vec{x} \cdot \vec{x} = 0$

$\therefore$ each term in $\vec{x}$, squared, then all terms added together, is 0.

This happens when $\vec{x} = \vec{0}$
(if and only if)

b.) (5 pts.) Show that for each vector $\mathbf{y}$ and each subspace W, the vector $(\mathbf{y} - \text{proj}_W \mathbf{y})$ is orthogonal to W.

By Orthogonal Decomposition Theorem,

$\vec{y} = \hat{y} + \vec{z}$ is a unique decomposition of $\vec{y}$ into a vector $\hat{y}$ in W and a vector $\vec{z}$ in $W^{\perp}$.

Then $\hat{y} = \text{proj}_W \vec{y}$ so $\vec{y} - \hat{y} = \vec{y} - \text{proj}_W \vec{y} = \vec{z}$, with $\vec{z}$ in $W^{\perp}$. Hence the vector is orthogonal to W. (since it is in $W^{\perp}$)

2.) (15 pts.) The following questions are all about eigenvalues and eigenvectors. Be sure to thoroughly explain each answer, and cite appropriate theorems when relevant.

a.) (5 pts.) Find the eigenvalues of the matrix $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 1 & 3 \end{bmatrix}$.

This is a triangular matrix, therefore the diagonal entries are eigenvalues: 5, 0, 3.

(Alternately: use the characteristic polynomial.)

b.) (5 pts.) Let A be a 3×3 matrix having three distinct eigenvalues. Explain how to diagonalize A.

Each eigenvalue will then have a 1-dimensional eigenspace, which can be represented using a single basis vector. If eigenvalues $\lambda_1, \lambda_2, \lambda_3$ have corresponding basis eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, then $A = PDP^{-1}$ where

$$P = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

c.) (5 pts.) Let $A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. Find a basis for the eigenspace corresponding to the eigenvalue $\lambda = 4$.

Solve $(A - 4I)\vec{x} = \vec{0}$: $\begin{bmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$

$(A-4I) \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\therefore \vec{x} = x_3 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = x_3 \vec{u} + x_4 \vec{v}$

Basis: $\{\vec{u}, \vec{v}\}$

3.) (15 pts.) The following questions are all about linear transformations. Be sure to thoroughly explain each answer, and cite appropriate theorems when relevant.

a.) (5 pts.) For the matrix $A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$, find all $\mathbf{x}$ in $\mathbb{R}^4$ that are mapped into the zero vector by the transformation $\mathbf{x} \to A\mathbf{x}$.

In other words: solve $A\vec{x} = \vec{0}$.

$$A \sim \begin{bmatrix} 1 & 0 & -9 & 7 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } \vec{x} = x_3 \begin{bmatrix} 9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Any $\vec{x}$ of this form is mapped to $\vec{0}$.

b.) (5 pts.) Show that the transformation T defined by $T(x_1, x_2) = (5x_2 - 3, 7x_1 + x_2)$ is not linear.

Linear: $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$, for any $\vec{u}, \vec{v}$.

Here: let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Then $T(\vec{u}) = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$, $T(\vec{v}) = \begin{bmatrix} 17 \\ 25 \end{bmatrix}$.

$\vec{u} + \vec{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ and $T(\vec{u} + \vec{v}) = \begin{bmatrix} 27 \\ 34 \end{bmatrix}$

But $T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} 24 \\ 34 \end{bmatrix} \neq T(\vec{u} + \vec{v})$. So: not linear.

c.) (5 pts.) Suppose A is $n \times n$ and the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Can you guarantee that A is invertible?

Yes. Since the linear transformation is onto part (i) of the IMT is true. Hence all parts are true, including (a): A is invertible.

4.) (15 pts.) Create or compute the following. Show your work carefully. Where creating something, be sure to show that your creation satisfies all the properties it needs to satisfy.

a.) (5 pts.) Create a matrix U for which all of the following three things are true: 1) U has at least two rows and at least two columns; 2) the columns of U form an orthonormal set; and 3) $U^TU \neq UU^T$.

Let $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$. Then 1.) U has 3 rows $(3 \geq 2)$ and 2 columns.

2.) Columns are $\vec{u}_1$ and $\vec{u}_2$, with $\vec{u}_1 \cdot \vec{u}_2 = 0$ → orthogonal.

Each column has length $\sqrt{1^2+0^2+0^2} = 1$.

So: orthonormal set.

3.) U^TU is 2×2, and UU^T is 3×3, so they must be different: $U^TU \neq UU^T$.

b.) (5 pts.) Let $\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of a vector in Span$\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.

$$\text{proj}_{\vec{u}} \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{14+6}{49+1} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}, \text{ in Span}\{\vec{u}\}$$

Vector orthogonal to $\vec{u}$: $\vec{y} - \text{proj}_{\vec{u}} \vec{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$

So $\vec{y} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$

c.) (5 pts.) Let $\mathbf{y} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}$, and $\mathbf{u}_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}$. Find the closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

Notice: $\vec{u}_1 \cdot \vec{u}_2 = 10 + 0 - 4 - 6 = 0$: orthogonal.

Now: $\hat{y} = \dfrac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \dfrac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2$

$$= \frac{4+0+0+3}{4+0+1+9} \vec{u}_1 + \frac{10-8+0-2}{25+4+16+4} \vec{u}_2$$

$$= \frac{1}{2} \vec{u}_1 \longrightarrow = \begin{bmatrix} 1 \\ 0 \\ -1/2 \\ -3/2 \end{bmatrix}$$

5.) (15 pts.) The following questions are all about matrices. Be sure to thoroughly explain each answer, and cite appropriate theorems when relevant.

a.) (5 pts.) Suppose an $n \times n$ matrix A is invertible. What is the simplest way to represent the inverse of A^{-1} ?

A $\qquad ((A^{-1})^{-1} = A)$

b.) (5 pts.) Rewrite the system of equations $x_1 \begin{bmatrix} -4 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ as a matrix equation.

$$\begin{bmatrix} -4 & 2 & 1 \\ 5 & -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

c.) (5 pts.) What is the maximum number of pivot positions possible in a 17×32 matrix? Why?

17 ∵ Can't have more than # of rows or # of columns, so whichever is the smallest number determines our limit.

6.) (15 pts.) The following questions are all about vector spaces. Be sure to thoroughly explain each answer, and cite appropriate theorems when relevant.

a.) (5 pts.) Let W be the set of all vectors of the form $\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$, where a and b represent arbitrary real numbers. Either find a set S of vectors that spans W or give an example to show that W is *not* a vector space.

We can't get the zero vector, so this is not a vector space.

To see this: to get $a - 6b = 0$, we would need $a = 6b$

Then $2b + a = 0$ means $2b + 6b = 0$, that is, $b = 0$.

Then $a = 0$. Then $-a+1 = 1$, not 0.

b.) (5 pts.) Some vector spaces are isomorphic. For example, we showed that $\mathbb{P}_3$ and $\mathbb{R}^4$ are isomorphic. Describe what it means for a pair of vector spaces to be isomorphic.

(Several answers)

c.) (5 pts.) Consider possible subspaces of $\mathbb{R}^3$ and their different dimensions. What are the possible dimensions of subspaces of $\mathbb{R}^3$? What geometric shape (or appearance) does each dimension of subspace have?

Point: $\vec{0}$ only, dimension 0.

Line: dimension 1. } through the origin, $\vec{0}$.

Plane: dimension 2. } through the origin, $\vec{0}$.

3-D space: dimension 3. (All of $\mathbb{R}^3$)

7.) (15 pts.) To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every two seconds, at times $t = 0, 2, 4, 6$ (in seconds). The positions (in feet) were: 0, 29.9, 104.7, 222.0.

a.) (5 pts.) Using linear algebra techniques, find the least-squares cubic curve $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$ for these data. Show all relevant steps.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \\ 1 & 6 & 36 & 216 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 29.9 \\ 104.7 \\ 222.0 \end{bmatrix} \quad \text{is } X\vec{\beta} = \vec{y}$$

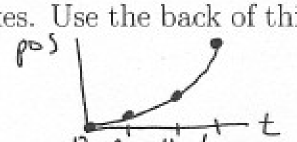
Solve: $X^T X \vec{\beta} = X^T \vec{y}$:

$$\begin{bmatrix} 4 & 12 & 56 & 288 \\ 12 & 56 & 288 & 1568 \\ 56 & 288 & 1568 & 8832 \\ 288 & 1568 & 8832 & 50816 \end{bmatrix} \vec{\beta} = \begin{bmatrix} 356.6 \\ 1810.6 \\ 9786.8 \\ 54892 \end{bmatrix}$$

Solving: $\vec{\beta} = \begin{bmatrix} 0 \\ 3.325 \\ 5.9125 \\ -.05 \end{bmatrix}$ So

$$y = 0 + 3.325t + 5.9125t^2 - .05t^3$$

b.) (5 pts.) Sketch the cubic curve you found in part (a), and the data points, on the same set of axes. Use the back of this page, or the back of the previous page, for your graph.



c.) (5 pts.) Use the result of part (a) to estimate the velocity of the plane when $t = 3.5$ seconds. Be sure to state how you are using your part (a) result to obtain this estimate.

In (a) we found position. Its derivative is velocity:

$v(t) = 3.325 + 11.825t - .15t^2$ At $t = 3.5$:

$v(3.5) = 42.875$