

NAME: \_\_\_\_\_

YOUR GRADE IS BASED ON CORRECTNESS, COMPLETENESS, AND CLARITY ON EACH EXERCISE. YOU MAY USE A CALCULATOR AND YOUR SINGLE SHEET OF PAPER WITH NOTES, BUT MAY NOT CONSULT BOOKS OR OTHER STUDENTS. GOOD LUCK!

1.) (10 pts.) Use theorems and concepts from Chapter 6 to address the following. Always explain your reasoning thoroughly.

a.) (5 pts.) Show that the only case in which a vector  $\mathbf{x}$  can be in both the subspaces  $W$  and  $W^\perp$  is when  $\mathbf{x} = \mathbf{0}$ .

b.) (5 pts.) Show that for each vector  $\mathbf{y}$  and each subspace  $W$ , the vector  $(\mathbf{y} - \text{proj}_W \mathbf{y})$  is orthogonal to  $W$ .

2.) (15 pts.) The following questions are all about eigenvalues and eigenvectors. Be sure to thoroughly explain each answer, and cite appropriate theorems when relevant.

a.) (5 pts.) Find the eigenvalues of the matrix  $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 1 & 3 \end{bmatrix}$ .

b.) (5 pts.) Let  $A$  be a  $3 \times 3$  matrix having three distinct eigenvalues. Explain how to diagonalize  $A$ .

c.) (5 pts.) Let  $A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ . Find a basis for the eigenspace corresponding to the eigenvalue  $\lambda = 4$ .

3.) (15 pts.) The following questions are all about linear transformations. Be sure to thoroughly explain each answer, and cite appropriate theorems when relevant.

a.) (5 pts.) For the matrix  $A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$ , find all  $\mathbf{x}$  in  $\mathbb{R}^4$  that are mapped into the zero vector by the transformation  $\mathbf{x} \rightarrow A\mathbf{x}$ .

b.) (5 pts.) Show that the transformation  $T$  defined by  $T(x_1, x_2) = (5x_2 - 3, 7x_1 + x_2)$  is not linear.

c.) (5 pts.) Suppose  $A$  is  $n \times n$  and the linear transformation  $T(\mathbf{x}) = A\mathbf{x}$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ . Can you guarantee that  $A$  is invertible?

4.) (15 pts.) Create or compute the following. Show your work carefully. Where creating something, be sure to show that your creation satisfies all the properties it needs to satisfy.

a.) (5 pts.) Create a matrix  $U$  for which all of the following three things are true: 1)  $U$  has at least two rows and at least two columns; 2) the columns of  $U$  form an orthonormal set; and 3)  $U^T U \neq U U^T$ .

b.) (5 pts.) Let  $\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$  and  $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ . Write  $\mathbf{y}$  as the sum of a vector in  $\text{Span}\{\mathbf{u}\}$  and a vector orthogonal to  $\mathbf{u}$ .

c.) (5 pts.) Let  $\mathbf{y} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}$ ,  $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}$ , and  $\mathbf{u}_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}$ . Find the closest point to  $\mathbf{y}$  in the subspace  $W$  spanned by  $\mathbf{u}_1$  and  $\mathbf{u}_2$ .

5.) (15 pts.) The following questions are all about matrices. Be sure to thoroughly explain each answer, and cite appropriate theorems when relevant.

a.) (5 pts.) Suppose an  $n \times n$  matrix  $A$  is invertible. What is the simplest way to represent the inverse of  $A^{-1}$ ?

b.) (5 pts.) Rewrite the system of equations  $x_1 \begin{bmatrix} -4 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$  as a matrix equation.

c.) (5 pts.) What is the maximum number of pivot positions possible in a  $17 \times 32$  matrix? Why?

6.) (15 pts.) The following questions are all about vector spaces. Be sure to thoroughly explain each answer, and cite appropriate theorems when relevant.

a.) (5 pts.) Let  $W$  be the set of all vectors of the form  $\begin{bmatrix} -a + 1 \\ a - 6b \\ 2b + a \end{bmatrix}$ , where  $a$  and  $b$  represent arbitrary real numbers. Either find a set  $S$  of vectors that spans  $W$  or give an example to show that  $W$  is *not* a vector space.

b.) (5 pts.) Some vector spaces are isomorphic. For example, we showed that  $\mathbb{P}_3$  and  $\mathbb{R}^4$  are isomorphic. Describe what it means for a pair of vector spaces to be isomorphic.

c.) (5 pts.) Consider possible subspaces of  $\mathbb{R}^3$  and their different dimensions. What are the possible dimensions of subspaces of  $\mathbb{R}^3$ ? What geometric shape (or appearance) does each dimension of subspace have?

7.) (15 pts.) To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every two seconds, at times  $t = 0, 2, 4, 6$  (in seconds). The positions (in feet) were: 0, 29.9, 104.7, 222.0.

a.) (5 pts.) Using linear algebra techniques, find the least-squares cubic curve  $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$  for these data. Show all relevant steps.

b.) (5 pts.) Sketch the cubic curve you found in part (a), and the data points, on the same set of axes. Use the back of this page, or the back of the previous page, for your graph.

c.) (5 pts.) Use the result of part (a) to estimate the velocity of the plane when  $t = 3.5$  seconds. Be sure to state how you are using your part (a) result to obtain this estimate.