1. Let $A = \begin{bmatrix} 9 & 2 & 1 \\ 1 & 3 & -1 \\ 7 & 1 & 1 \end{bmatrix}$ and label its columns $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$. Let $\mathbf{d} = \begin{bmatrix} 7 \\ 5 \\ 13 \end{bmatrix}$.

1A. Explain why the matrix equation $A\mathbf{x} = \mathbf{d}$ is inconsistent.

1B. Since $\mathbf{d}$ is not in Col A, find the least squares solutions to $A\mathbf{x} = \mathbf{d}$. Show all matrices involved.

1C. From the answer to 1B, you can determine the projection vector $\mathbf{w}$ of $\mathbf{d}$ onto Col A. Find $\mathbf{w}$.

1D. Explain how you can tell that the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ form a basis of Col A.

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1E. It turns out that $\mathbf{a}_2$ and $\mathbf{a}_3$ also form a basis for Col A; you don't need to show this. But do show that $\mathcal{C} = \{\mathbf{a}_2, \mathbf{a}_3\}$ is an *orthogonal* basis.

1F. Since $\mathcal{C}$ is an orthogonal basis, there's a formula using dot products that you can use to find the projection vector $\mathbf{w}$ of $\mathbf{d}$ onto Col A. Give that formula *and* evaluate it (show all the work) and verify it produces the same projection vector $\mathbf{w}$ as in 1C.

1G. Find the distance from $\mathbf{d}$ to Col A. Show all your work. Give a decimal approximation of your answer to two decimal places.

2. Consider the matrices M and N:

$$\text{Let } M = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 8 & 6 \\ 1 & 1 & -1 & 1 \\ 1 & 3 & 9 & 5 \end{bmatrix}; \text{ the RREF of } \left[\begin{array}{cccc|cccc} 1 & 2 & 4 & 3 & 1 & 0 & 0 & 0 \\ 2 & 4 & 8 & 6 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 3 & 9 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \text{ is } \left[\begin{array}{cccc|cccc} 1 & 0 & -6 & -1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 5 & 2 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right]$$

$$\text{Let } N = \begin{bmatrix} 4 & 3 & 1 & 2 \\ 8 & 6 & 2 & 4 \\ -1 & 1 & 1 & 1 \\ 9 & 5 & 1 & 3 \end{bmatrix}; \text{ the RREF of } \left[\begin{array}{cccc|cccc} 4 & 3 & 1 & 2 & 1 & 0 & 0 & 0 \\ 8 & 6 & 2 & 4 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 9 & 5 & 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \text{ is } \left[\begin{array}{cccc|cccc} 1 & 0 & -2/7 & -1/7 & 0 & 0 & -5/14 & 1/14 \\ 0 & 1 & 5/7 & 6/7 & 0 & 0 & 9/14 & 1/14 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right]$$

Label the columns of M as $\mathbf{m}_1,\ldots,\mathbf{m}_4$; then note that N is just $[\mathbf{m}_3\ \ \mathbf{m}_4\ \ \mathbf{m}_1\ \ \mathbf{m}_2]$ and so Col M and Col N are equal.

From the RREF's above, you can tell that each of $\mathcal{B} = \{\mathbf{m}_1, \mathbf{m}_2\}$ and $\mathcal{C} = \{\mathbf{m}_3, \mathbf{m}_4\}$ is a basis for Col M (and so for Col N as well since Col M = Col N).

2A. There's enough information above for you to find $[\mathbf{m}_1]_{\mathcal{C}}$ and $[\mathbf{m}_3]_{\mathcal{B}}$. Find them both.

2B. Find the Change of Basis matrix P from $\mathcal{B}$ to $\mathcal{C}$. (one of the answers in 2A is useful here: be careful!)

2C. Suppose that for some vector $\mathbf{b}$ in Col M, that $[\mathbf{b}]_{\mathcal{B}}$ is $\begin{bmatrix} -7 \\ 6 \end{bmatrix}$. Use P to find $[\mathbf{b}]_{\mathcal{C}}$.

2D. Explicitly find the vector $\mathbf{b}$. (Hint: use $[\mathbf{b}]_{\mathcal{B}} = \begin{bmatrix} -7 \\ 6 \end{bmatrix}$).

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2. continued For your reference, here is again is the information about the matrix M:

Let $M = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 8 & 6 \\ 1 & 1 & -1 & 1 \\ 1 & 3 & 9 & 5 \end{bmatrix}$; the RREF of $\left[\begin{array}{cccc|cccc} 1 & 2 & 4 & 3 & 1 & 0 & 0 & 0 \\ 2 & 4 & 8 & 6 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 3 & 9 & 5 & 0 & 0 & 0 & 1 \end{array}\right]$ is $\left[\begin{array}{cccc|cccc} 1 & 0 & -6 & -1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 5 & 2 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right]$

2E. What conditions on $d_1,\ldots,d_4$ must be satisfied for $\mathbf{d} = [d_1 \;\; d_2 \;\; d_3 \;\; d_4]^T$ to be in Col M? (remember we're just using the transpose notation so we can write this vector sideways to save space)

2F. Verify that $\mathbf{m}_1$ satisfies the conditions in 2E.

2G. (BONUS) Those conditions in 2E allow you to find another basis for Col M. Find that basis.

2H. Use the RREF of M to find a basis of Row M.

2I. Show how to express the last row of M as a linear combination of the basis vectors in 2H.

2J. Find a basis for Nul M.

3. Let $A = \begin{bmatrix} 7 & -2 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

3A. Find the characteristic polynomial of A; show all your work. You should find there are two eigenvalues. What are they?

3B. Calculate $A \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and you will discover an eigenvector for one of your eigenvalues. What is that eigenvalue?

3C. Find a basis for the eigenspace corresponding to the *other* eigenvalue (that is, NOT the one in 3B)

3D. Is the matrix A diagonalizable? If not, explain why not. Otherwise give a pair of matrices P and D with the required properties.

4. The points (1,5) and (2,7) are on the line $y = 3+2x$, while the points (4,7) and (5,9) are on the line $y = -1+2x$; this is easy to check. Since the four points are on different lines, there's no line of the form $y = \beta_0 + \beta_1 x$ which contains these points.

4A. Completely set up and solve the problem of finding the best-fit line through these four points. Express your answers as decimals (not fractions).

4B. What is the sum-of-squares (SOS) of the residuals for this best-fit line?

4C. Those two lines $y = 3 + 2x$ and $y = -1 + 2x$ are parallel since they have the same slope. So you might think the line in the "middle" of them, $y = 1 + 2x$, would be the best fit line. Show this is not the case by finding the SOS for THIS line.

4D. What is the design matrix X you would set up in order to find the best-fit parabola of the form $y = \beta_0 + \beta_1 x + \beta_2 x^2$ for these same four points?

NOTE: Do *either* problem 5 *or* problem 6; it's your choice. CROSS OUT the one you do NOT do.

5. Suppose that $\mathbb{F}$ is the space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. Let H be the subset of $\mathbb{F}$ of all functions $g \in \mathbb{F}$ satisfying $g(2) = g(4)$. That is, a g function is in H iff it has the same y coordinate at $x = 2$ as it does at $x = 4$. Prove that H satisfies the "addition" part of the subspace definition (ie, that H is closed under addition), or give an explicit counterexample to show otherwise.

NOTE: Do *either* problem 5 *or* problem 6; it's your choice. CROSS OUT the one you do NOT do.

6. Suppose we define $T : \mathbb{P}_2 \to \mathbb{R}^3$ by $T(ax^2 + bx + c) = \begin{bmatrix} a+c \\ 0 \\ b \end{bmatrix}$.

6A. Prove that T satisfies the "addition" part of the definition of linear transformation (that is, the part that says $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{P}_2$).

6B. Show that T is not 1-1 by giving an explicit counterexample.

6C. Show that T is not onto by giving an explicit counterexample.

7. Answer each of the following: If the answer is "there is none" or "does not exist" (etc) explain why.

7A: The standard basis for $\mathbb{R}^3$ is?

7B: Give a basis for $\mathbb{P}_3$.

7C: Can 0 be an eigenvalue of a matrix A? Give an example or explain why not.

7D: Suppose that E_3, E_2 and E_1 are elementary matrices and $E_3E_2E_1A = I_4$. Express A^{-1} as a product of the inverses of the E_i's.

7E. What is the dimension of $\mathbb{P}_2$?

7F. What is a basis for the subspace $H = \{\mathbf{0}\}$ in any vector space V?

7G. Give an example of an infinite dimensional vector space.

7H. What is the dimension of $H = \{\mathbf{0}\}$?