Math 106A - Final Exam - April 7, 2006

INSTRUCTIONS: Show all of your work and circle your solutions. Cross out any unnecessary work.

1. (7 pts. each) Calculate the following:

(a) \[ \int \frac{e^x}{e^x + 1} \, dx. \]

Let \( u = e^x + 1 \), so \( du = e^x \, dx \).

\[ = \int \frac{du}{u} = \ln |u| + c = \ln |e^x + 1| + c. \]

(b) \[ \int_0^\pi x \cos x \, dx. \]

Int. by parts.

\[ u = x \quad \quad du = dx \]
\[ dv = \cos x \, dx \quad v = \sin x. \]

\[ = x \sin x \bigg|_0^\pi - \int_0^\pi \sin x \, dx \]

\[ = (\pi \sin \pi - 0) - \left( -\cos x \right)_0^\pi \]
\[ = (0 - 0) - (-\cos \pi - (-\cos 0)) \]
\[ = 1 - 1 = -2 \]
(e) \[ \int_{2}^{\infty} x e^{-x^2} \, dx. \]

\[ = \lim_{t \to \infty} t \int_{2}^{t} x e^{-x^2} \, dx. \quad u = -x^2, \; du = -2x \, dx. \quad u(2) = -4, \; u(t) = -t^2. \]

\[ = \lim_{t \to \infty} -\frac{1}{2} e^{-u} \bigg|_{2}^{t} \]

\[ = \lim_{t \to \infty} -\frac{1}{2} e^{-t^2} - \frac{1}{2} e^{-4} \]

\[ = \lim_{t \to \infty} -\frac{1}{2} e^{-t^2} + \frac{1}{2} e^{-4} \]

\[ = -0 + \frac{1}{2 e^4} = \frac{1}{2e^4} \]

2. (8 pts.) Consider the region enclosed by the graphs of \( x = 0, \; y = 0, \; x = 2, \) and \( y = x^2. \) Set up an integral (but do not solve) that gives the volume of the solid obtained by rotating this region about the line \( x = -3. \) (With a clear picture, explain where the terms in your integral come from.)

Area of washer = \( \pi (5)^2 - \pi (x+3)^2 \)

Thickness = \( \Delta y. \)

Volume of washer = \( (25 \pi - \pi (x+3)^2) \, \Delta y. \)

So total volume = \( \int_{y=0}^{y=4} (25 \pi - \pi (y+3)^2) \, dy. \)
3. (7 pts.) Calculate the exact value of \(96 - 48 + 24 - 12 + 6 - 3 + \cdots\) or explain why it doesn't exist.

Geometric sum with \(a = 96\), \(r = -\frac{1}{2}\).

Since \(|r| < 1\), so the series converges to \(a = \frac{96}{1 - (-\frac{1}{2})} = \frac{96}{\frac{3}{2}} = 64\).

4. (7 pts. each) Determine if the following series are convergent or divergent. Explain your answer and make sure to mention which convergence test you’re using.

(a) \(\sum_{n=1}^{\infty} \frac{n(n-1)}{3n^2 + 4}\)

Let \(a_n = \frac{n(n-1)}{3n^2 + 4} = \frac{n^2 - n}{3n^2 + 4} = \frac{1 - \frac{1}{n}}{3 + \frac{4}{n^2}}\).

As \(n \to \infty\), \(a_n \to \frac{1}{3} \neq 0\), so by the nth term test, since the terms of the series do not converge to zero, the series is divergent.

(b) \(\sum_{k=0}^{\infty} \frac{k!}{(2k)!}\)

Ratio test.

\(a_n = \frac{k!}{(2k)!}\) \(a_{n+1} = \frac{(k+1)!}{(2(k+1))!} = \frac{(k+1)!}{(2k+2)!} = \frac{k!(k+1)}{(2k)!(2k+1)(2k+2)}\)

\(\frac{a_{n+1}}{a_n} = \frac{k!(k+1)}{(2k)!(2k+1)(2k+2)} \cdot \frac{(2k)!}{k!} = \frac{k+1}{(2k+1)(2k+2)} = \frac{1}{4k+2}\).

So \(\lim_{k \to \infty} \frac{a_{n+1}}{a_n} = \lim_{k \to \infty} \frac{1}{4k+2} = 0\). Thus, since this limit is less than 1, by ratio test, the series converges.
5. (9 pts.) The following is a convergent alternating series.

\[ S = \sum_{j=1}^{\infty} \frac{(-1)^j}{\sqrt{j}} \]

(a) Is this series absolutely convergent or conditionally convergent? Explain and mention the convergence test you're using.

\[ \sum_{j=1}^{\infty} \left| \frac{(-1)^j}{\sqrt{j}} \right| = \sum_{j=1}^{\infty} \frac{1}{\sqrt{j}} \] which diverges by \( p \)-series test. \( p = \frac{1}{2} < 1 \).

So \( S \) is a conditionally convergent series.

(b) In order to estimate \( S \) with error at most \( \frac{1}{100} \), how many terms in the series would you have to sum together? (You do not need to calculate this partial sum.)

\[ |S - S_m| < C_n \] for the series \( \sum (-1)^n c_n \), with \( c_1 > c_2 > \cdots > 0 \).

So we need \( |S - S_m| < \frac{1}{100} \).

\[ c_n = \frac{1}{n^{1/2}} < \frac{1}{100} \]

\[ n > 1000000 \]. So \( S_n \), for \( n > 999999 \), will give the desired accuracy.

6. (6 pts.) Let \( g(x) = \frac{x}{1 - 2x^3} \).

(a) Find the Maclaurin series for \( g(x) \).

(b) Calculate \( g^{(10)}(0) \).

a)

\[ \frac{1}{1-x} = 1 + x + x^2 + \cdots \quad \text{for } |x| < 1. \]

\[ \frac{1}{1-2x^3} = 1 + (2x^3) + (2x^3)^2 + (2x^3)^3 + \cdots \]

\[ = 1 + 2x^3 + 4x^6 + 8x^9 + \cdots \]

b)

\[ g^{(10)}(0) = 10! \cdot a_{10} = \frac{10!}{8}. \]
7. (10 pts.) Find the radius and interval of convergence (check endpoints!) of the following power series:

\[ \sum_{k=1}^{\infty} \frac{x^k}{3^k k^{1/2}} \]

**Ratio test:**

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{3^{n+1} (n+1)^{1/2}} \right| = \frac{\frac{1}{3}}{\frac{1}{\sqrt{n}}} = \frac{\sqrt{n}}{3} = \frac{\sqrt{1}}{3} = 1.
\]

By ratio test, the series converges if \( |\frac{x}{3}| < 1 \), so if \( |x| < 3 \).

And the series diverges if \( |\frac{x}{3}| > 1 \), so if \( |x| > 3 \).

We therefore check the endpoints \( x = 3 \) and \( x = -3 \).

**\( x = 3 \):**

\[ \sum_{k=1}^{\infty} \frac{x^k}{3^k k^{1/2}} = \sum_{k=1}^{\infty} \frac{3^k}{3^k} = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}, \text{ divergent by p-series} \quad (p = \frac{1}{2} < 1). \]

**\( x = -3 \):**

\[ \sum_{k=1}^{\infty} \frac{(-3)^k}{3^k k^{1/2}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/2}} \text{ alternating series with } C_k = \frac{1}{\sqrt{k}}. \]

We have 1) \( \lim_{k \to \infty} C_k = \lim_{k \to \infty} \frac{1}{\sqrt{k}} = 0 \)

2) \( \frac{1}{\sqrt{1}} > \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} > \frac{1}{\sqrt{4}} > \cdots > 0. \)

So this is a convergent alternating series.

**Hence, the interval of convergence is \([ -3, 3 )\).**

And the radius of convergence is 3.
Let \( f(x) = \ln(1 + 2x) \).

(a) Find the degree-2 Taylor polynomial of \( f(x) \) at \( x_0 = 0 \).

\[
\begin{align*}
\hat{f}(x) &= \frac{1}{1+2x} \\
\hat{f}'(x) &= -\frac{2}{(1+2x)^2} \\
\hat{f}''(x) &= \frac{4}{(1+2x)^3} \\
\hat{f}'''(x) &= \frac{24}{(1+2x)^4} \\
f(0) &= 0 \\
f'(0) &= \frac{2}{1+0} = 2 \\
f''(0) &= \frac{-4}{(1+0)^2} = -4 \\
f'''(0) &= \frac{-4}{(1+0)^3} = -2
\end{align*}
\]

The Taylor polynomial of degree two is:

\[
P_2(x) = \hat{f}(x_0) + \frac{\hat{f}'(x_0)}{1!}(x-x_0) + \frac{\hat{f}''(x_0)}{2!}(x-x_0)^2
\]

\[
= 0 + 2(x-0) + \frac{-4}{2} (x-0)^2
\]

\[
= 2x - 2x^2.
\]

(b) Find an approximate value of \( \ln 1.5 \).

\[
X = 0.25 = \frac{1}{4}.
\]

\[
P_2 \left( \frac{1}{4} \right) = 2 \left( \frac{1}{4} \right) - 2 \left( \frac{1}{4} \right)^2
\]

\[
= \frac{1}{2} - \frac{1}{8} = \frac{3}{8} = 0.375.
\]

(c) Using Taylor's Theorem, find the maximum possible error in approximating \( \ln 1.5 \).

\[
\left| f(x) - P_n(x) \right| \leq \frac{M_{n+1}}{(n+1)!} |x-x_0|^{n+1}
\]

\[n = 2, \quad x_0 = 0, \quad x = \frac{1}{4} \]

\[
f''(x) = \frac{16}{(1+2x)^3}
\]

As \( x \) increases on the interval \([0, 1/4]\), the denominator of
\[f''(x)\]
decreases, so \( |f''(x)| \) increases, so \( |f''(x)| \)
is maximal at \( x=0 \).

\[K = \frac{16}{1^3} = 16.
\]

\[
\left| f(x) - P_2(x) \right| \leq \frac{16}{3!} \left| 0 - \frac{1}{4} \right|^3 = \frac{1}{24} \approx 0.4166.
\]
9. (5 pts. each) For each of the following sequences \( \{a_k\}_{k=1}^{\infty} \), determine, with proper justification, whether the sequence is convergent or divergent. If convergent, find the value that the sequence converges to.

(a) \( a_k = ke^{-k} \)

\[
\lim_{k \to \infty} f(x) = x e^{-x} = \frac{x}{e^x}, \quad \lim_{k \to \infty} f(x) = \lim_{k \to \infty} \frac{x}{e^x} = \frac{\infty}{\infty}
\]

By l'Hôpital's rule, \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{e^x} = 0 \).

Since \( a_k = f(k) \) and \( \lim_{k \to \infty} f(k) = 0 \), \( \lim_{k \to \infty} a_k = 0 \).

(b) \( a_k = \cos \left( \frac{2}{k} \right) \)

\[
f(x) = \cos \left( \frac{2}{x} \right), \quad \lim_{x \to \infty} \cos \left( \frac{2}{x} \right) = \cos \left( \frac{2}{\infty} \right) = \cos(0) = 1.
\]

\[
\lim_{x \to \infty} \frac{2}{x} = 0 \), so \( \lim_{x \to \infty} \cos \left( \frac{2}{x} \right) = \cos(0) = 1.
\]

Hence, \( \lim_{k \to \infty} a_k = 1 \).

10. (5 pts.) Using power series (instead of l'Hôpital's rule), find \( \lim_{x \to 0} \frac{2x - \sin(2x)}{x^3} \).

\[
\sin(2x) = 2x - \left( \frac{2x}{3!} \right)^3 + \left( \frac{2x}{5!} \right)^5 - \ldots = 2x - \frac{8x^3}{6} + \frac{32x^5}{5!} - \ldots
\]

\[
\frac{2x - \sin 2x}{x^3} = \frac{1}{x^3} \left( 2x - \left( 2x - \left( 2x - \left( \frac{8x^3}{6} + \frac{32x^5}{5!} + \ldots \right) \right) \right) \right)
\]

\[
= \frac{1}{x^3} \left( \frac{8x^3}{3} - \frac{32x^5}{5!} + \ldots \right)
\]

\[
= \frac{4}{3} - \frac{32x^2}{5!} + \ldots
\]

So \( \lim_{x \to 0} \frac{2x - \sin 2x}{x^3} = \frac{4}{3} - 0 + 0 + \ldots = \frac{4}{3} \).

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