1. (14 points) Evaluate the following integrals exactly.

(a) \[ \int \frac{6x - 20}{4x - x^2} \, dx \]

\[ 4x - x^2 = x(4-x) \]

\[ \frac{6x - 20}{x(4-x)} = \frac{A}{x} + \frac{B}{4-x} \]

\[ 6x - 20 = A(4-x) + B(x) \]

Setting \( x = 4 \), we get \( 4 = 4B \). So \( B = 1 \).

Setting \( x = 0 \), we get \( -20 = 4A \). So \( A = -5 \).

\[ \int \frac{6x - 20}{4x - x^2} \, dx = \int \frac{-5}{x} \, dx + \int \frac{1}{4-x} \, dx \]

(For the second integral, substitute \( u = 4-x, \, du = -dx \).

\[ = -5 \ln |x| + \ln |4-x| + C. \]

(b) \[ \int \frac{x^2}{\sqrt{25 + x^2}} \, dx \]

\( x = 5 \tan t, \, dx = 5 \sec^2 t \, dt \)

\[ = \int \frac{25 \tan^2 t \cdot 5 \sec^2 t \, dt}{\sqrt{25 + 25 \tan^2 t}} \]

\[ = \int \frac{25 \tan^2 t \cdot \sec^2 t \, dt}{\frac{5}{\sqrt{1 + \tan^2 t}}} \]

\[ = \int \frac{25 \tan^2 t \cdot \sec^2 t \, dt}{\sec t} \]

\[ = 25 \int \tan^2 t \cdot \sec t \, dt = 25 \int \sec t \cdot \tan^2 t \, dt \]

\[ = 25 \int (\sec^3 t - \sec t) \, dt \]

\[ = 25 \int \sec^3 t \, dt - 25 \int \sec t \, dt \]

\[ = 25 \left( \frac{\sec t \cdot \tan t}{2} + \frac{1}{2} \int \sec t \, dt \right) - 25 \int \sec t \, dt \quad \text{(Formula #51)} \]

\[ = \frac{25}{2} \sec t \cdot \tan t - 25 \int \sec t \, dt \]

\[ = \frac{25}{2} \sec t \cdot \tan t - 25 \ln |\sec t + \tan t| + C \quad \text{(Formula #17)} \]

\[ \tan t = \frac{x}{5} \]

\[ \sec t = \frac{\sqrt{25 + x^2}}{5} \]

\[ \frac{\sqrt{25 + x^2}}{5} \]
2. (6 points) Find the solution of the IVP: \( e^x y' = xy^2, \ y(0) = 4 \).

\[
\begin{align*}
\frac{d}{dx} e^x &= x y^2 \\
\frac{dy}{y^2} &= x e^{-x} dx \\
\int y^{-2} dy &= \int x e^{-x} dx \\
y &= \frac{-1}{x e^{-x} + e^{-x} - \frac{3}{4}}.
\end{align*}
\]

Integration by parts for \( \int x e^{-x} dx \):
- \( u = x \) \quad \( dv = e^{-x} dx \)
- \( du = dx \) \quad \( v = -e^{-x} \)
- \( \int x e^{-x} dx = -xe^{-x} - \int e^{-x} dx = -xe^{-x} - e^{-x} \)
- \( \frac{1}{y} = -xe^{-x} - e^{-x} + C \)
- \( i.e. \frac{1}{y} = xe^x + e^{-x} - C \)
- \( y = \frac{1}{xe^x + e^{-x} - C} \)
- \( x = 0, y = 4 \) gives \( C = \frac{3}{4} \).

3. (5 points) Let \( I = \int_0^3 f(x) \, dx \). What is the least value of \( n \) which guarantees that \( T_n \) approximates \( I \) within \( \pm 0.001 \)? Justify your answer. The graph of \( |f''(x)| \) is given below.

We know
\[
|I - T_n| \leq \frac{K_2 (b-a)^3}{12 n^2}
\]
where \( K_2 = \max_{[0,3]} |f''(x)| \)

From the graph, \( K_2 = 4 \).

So \( 4 (3-0)^3 \leq 0.001 \)

\[
\begin{align*}
\frac{9}{n^2} &\leq 0.001 \\
n^2 &\geq \frac{9}{0.001} = 9000 \\
n &\geq \sqrt{9000} = 94.8683 \\
\text{So } n > 95 & \quad \text{Least value of } n = 95.
\end{align*}
\]
4. (6 points) Consider the IVP $y' = x - y$, $y(2.5) = 2$. Use Euler's method with two steps to estimate $y(3.5)$. (Do not use a calculator program for this problem.)

$$\Delta x = \frac{3.5 - 2.5}{2} = \frac{1}{2} = 0.5$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$(x-y) \Delta x = \Delta y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>2</td>
<td>$(2.5 - 2)(0.5) = 0.25$</td>
</tr>
<tr>
<td>3.0</td>
<td>2.25</td>
<td>$(3 - 2.25)(0.5) = 0.375$</td>
</tr>
<tr>
<td>3.5</td>
<td>2.625</td>
<td></td>
</tr>
</tbody>
</table>

$y(3.5) \approx 2.625$

5. (6 points) Consider the region bounded by the curve $y = e^x$, and the lines $x = 0$, $x = 1$ and the $x$-axis. Write (but do not evaluate) an integral to find the volume of the solid that is formed when the region is rotated about the line $y = e$.

Slice is a washer.

Volume of slice = $(\pi r_{out}^2 - \pi r_{in}^2) \Delta x$

$r_{out} = e$, $r_{in} = e - y = e - e^x$

So volume of slice = $(\pi e^2 - \pi (e - e^x)^2) \Delta x$

Total volume = $\int_0^1 (\pi (e^2 - (e - e^x)^2)) \, dx$
6. (6 points) A water tank is in the shape of a hemisphere of radius 10 feet and is filled with water to a height of 7 feet. The figure below shows the water tank. Write (but do not evaluate) an integral equal to the work done in pumping all the water over the top of the tank. Water weighs 62.4 lb per cubic foot.

\[
\text{Volume of water in slice} = \pi r^2 \Delta y
\]

\[
r = \sqrt{100 - y^2}
\]

\[
\text{Volume of water in slice} = \pi (100 - y^2) \Delta y \text{ ft}^3
\]

\[
\text{Weight of water in slice} = \pi (100 - y^2) \Delta y \cdot 62.4 \text{ lb}
\]

\[
\text{Work done on slice} = \pi (100 - y^2) \Delta y \cdot 62.4 \cdot (10 - y) \text{ lb-ft}
\]

\[
\text{Total work done} = \int_0^7 \pi (100 - y^2) (62.4)(10 - y) \, dy
\]

7. (6 points) Use comparisons to determine the convergence of the following series.

\[
\sum_{n=2}^{\infty} \frac{2n^3 - 3}{7 + 5n^2 + n^5}.
\]

\[
0 \leq \frac{2n^3 - 3}{7 + 5n^2 + n^5} \leq \frac{2n^3}{n^5} \text{ smaller denominator, larger fraction}
\]

\[
\sum_{n=2}^{\infty} \frac{2n^3}{n^5} = \sum_{n=2}^{\infty} \frac{2}{n^2}
\]

Since \( \sum_{n=2}^{\infty} \frac{1}{n^2} \) directly converges (\( p > 1 \)), \( \sum_{n=2}^{\infty} \frac{2}{n^2} \) converges.

So by comparison test, \( \sum_{n=2}^{\infty} \frac{2n^3 - 3}{7 + 5n^2 + n^5} \) converges.
8. (15 points) Determine the convergence of the following series. If possible, find the sum of the series if it converges.

(a) \( \frac{7}{2} + \frac{7}{4} + \frac{7}{8} + \frac{7}{16} + \ldots \) Geometric series with \( a = \frac{7}{2}, \ r = \frac{1}{2} \).

Since \(-1 < r < 1\), series converges.

\[
\text{Sum} = \frac{a}{1-r} = \frac{\frac{7}{2}}{1-\frac{1}{2}} = 7.
\]

(b) \( \frac{41}{10} + \frac{401}{100} + \frac{4001}{1000} + \frac{40001}{10000} + \ldots \)

\[
= \sum_{n=1}^{\infty} \frac{4(10^n)+1}{10^n} = \sum_{n=1}^{\infty} \left( \frac{4 + \frac{1}{10^n}}{10^n} \right)
\]

\[
\lim_{n \to \infty} \frac{4 + \frac{1}{10^n}}{10^n} = 4 + 0 = 4 \neq 0.
\]

Since so by the nth term test, series diverges.

(c) \( \sum_{n=2}^{\infty} \frac{\ln n}{n} \)

Use Integral Test.

The fn. \( y = \frac{\ln x}{x} \) is positive and decreasing on \([2, \infty)\).

\[
\int_{2}^{\infty} \frac{\ln x}{x} \, dx = \lim_{b \to \infty} \int_{2}^{b} \frac{u^2}{2} \, du = \lim_{b \to \infty} \left[ \frac{(\ln x)^2}{2} \right]_{x=2}^{b}
\]

\[
= \lim_{b \to \infty} \left( \frac{(\ln b)^2}{2} - \frac{(\ln 2)^2}{2} \right)
\]

\[
= \infty.
\]

So integral diverges. By integral test, the series diverges.
9. (8 points) Find the interval and radius of convergence of the series \( \sum_{n=2}^{\infty} \frac{(x-4)^n}{n3^n} \).

**Ratio Test:**

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x-4)^{n+1}}{(n+1)3^{n+1}} \right| = \frac{n}{n+1} \cdot \frac{|x-4|}{3}
\]

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{n}{n+1} \cdot \frac{|x-4|}{3} = \frac{|x-4|}{3}
\]

So by ratio test, series converges if \( \frac{|x-4|}{3} < 1 \) i.e. if \(|x-4| < 3\).

i.e. if \(-3 < x-4 < 3\)  i.e. if \(1 < x < 7\).

**Check endpoints:**

\(x = 1:\)

\[
\sum_{n=2}^{\infty} \frac{(1-4)^n}{n3^n} = \sum_{n=2}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n},
\]

which converges by the alternating series test.

\[
\left( \lim_{n \to \infty} \frac{1}{n} = 0, \quad 0 < \frac{1}{n} < \frac{1}{n+1} \quad \text{for all} \quad n \right)
\]

\(x = 7:\)

\[
\sum_{n=2}^{\infty} \frac{(7-4)^n}{n3^n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n},
\]

which diverges.

So the interval of convergence is \([1, 7]\).

Radius of convergence is 3.
10. (10 points) Let \( f(x) = \cos(2x^2) \). Use this function to answer the following questions. In this problem, you do not need to write any series using summation notation.

(a) Use a known power series to write the first four non-zero terms of the power series representation for \( f \).

We know \( \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \)

So \( \cos(2x^2) = 1 - \frac{(2x^2)^2}{2!} + \frac{(2x^2)^4}{4!} - \frac{(2x^2)^6}{6!} + \cdots \)

\( = 1 - \frac{4x^4}{2!} + \frac{16x^8}{4!} - \frac{64x^{12}}{6!} + \cdots \)

\( = 1 - 2x^4 + \frac{2}{3}x^8 - \frac{4}{45}x^{12} + \cdots \)

(b) Use your answer in part (a) to write the power series representation of \( \int_0^1 \cos(2x^2) \, dx \).

\[ \int_0^1 \cos(2x^2) \, dx = \int \left( 1 - 2x^4 + \frac{2}{3}x^8 - \frac{4}{45}x^{12} + \cdots \right) \, dx \]

\[ = x - 2x^5 + \frac{2}{3}x^9 - \frac{4}{45}x^{13} + \cdots \]

\[ \int_0^1 \cos(2x^2) \, dx = \left. x - 2x^5 + \frac{2}{3}x^9 - \frac{4}{45}x^{13} + \cdots \right|_0^1 \]

\[ = 1 - 2 + \frac{2}{5} - \frac{2}{3} + \frac{4}{45} + \cdots \]

(c) Use your answer in part (a) to evaluate the limit: \( \lim_{x \to 0} \frac{\cos(2x^2) - 1}{x^4} \).

\[ \lim_{x \to 0} \frac{\cos(2x^2) - 1}{x^4} = \lim_{x \to 0} \frac{\left( 1 - 2x^4 + \frac{2}{3}x^8 - \frac{4}{45}x^{12} + \cdots \right) - 1}{x^4} \]

\[ = \lim_{x \to 0} \frac{-2x^4 + \frac{2}{3}x^8 - \frac{4}{45}x^{12} + \cdots}{x^4} \]

\[ = \lim_{x \to 0} \left( -2 + \frac{2}{3}x^4 - \frac{4}{45}x^8 + \text{terms with } x \text{ in them} \right) \]

\[ = -2 + 0 = -2 \]
11. (8 points) Let \( f(x) = \ln x \).

(a) Find the first four non-zero terms of the Taylor series for \( f \) based at \( x_0 = 1 \). Then write the series using the summation notation.

\[
\begin{align*}
  f(x) &= \ln x, \quad f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}, \quad f'''(x) = \frac{2}{x^3}, \quad f^{(4)}(x) = -\frac{6}{x^4} \\
  f(1) &= 0, \quad f'(1) = 1, \quad f''(1) = -1, \quad f'''(1) = 2, \quad f^{(4)}(1) = -6 \\

  \text{Taylor series for } f \text{ based at } x_0 = 1 \text{ is} \\
  &1. \frac{(x-1) - \frac{1}{2} (x-1)^2 + \frac{2}{3} (x-1)^3 - \frac{6}{4!} (x-1)^4}{2!} = \frac{(x-1) - \frac{1}{2} (x-1)^2 + \frac{1}{3} (x-1)^3 - \frac{1}{4} (x-1)^4}{2} \\
  &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{(x-1)^n}{n}.
\end{align*}
\]

(b) Let \( P_3(x) \) be the third-order Taylor polynomial for \( f \) based at \( x_0 = 1 \). What does Taylor’s theorem imply about the maximum approximation error committed by \( P_3 \) over the interval \([0.5, 3]\)? (Find the best possible value for \( K_{n+1} \).)

\[
|f(x) - P_3(x)| \leq \frac{K_4}{4!} |x-1|^{4} \quad \text{(by Taylor’s theorem)}
\]

\[
K_4 = \max \{ |f^{(4)}(x)| \} \text{ on } [0.5, 3].
\]

\[
|f^{(4)}(0.5)| = \left| \frac{-6}{(0.5)^4} \right| = 96.
\]

\[
\text{Graph of } |f^{(4)}(x)| = \left| \frac{-6}{x^4} \right|.
\]

\[
|f(x) - P_3(x)| \leq \frac{96}{24} |x-1|^{4} \leq 4 |x-1|^{4}.
\]

On \([0.5, 3]\), \( (x-1)^4 \leq 13-1\frac{1}{4} = 2^4 = 16 \).

So \( |f(x) - P_3(x)| \leq 4 \cdot (16) \) on \([0.5, 3]\).

So by Taylor’s thm, the max. approximation error on \([0.5, 3]\) is \( 64 \).
12. (10 points) Suppose that the partial sums of the series \( \sum_{k=1}^{\infty} a_k \) are \( S_n = \sum_{k=1}^{n} a_k = \frac{2n + 5}{3n - 1} \).

(a) Does the sequence \( S_n \) converge? Justify your answer.

\[
\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{2n + 5}{3n - 1} = \lim_{n \to \infty} \frac{2}{3} \quad (L'Hôpital's \ rule)
\]

\[
\lim_{n \to \infty} S_n = \frac{2}{3}
\]

So the seq. \( S_n \) converges.

(b) Does the series \( \sum_{k=1}^{\infty} a_k \) converge? Justify your answer. If the series converges, what is its sum?

The series \( \sum_{k=1}^{\infty} a_k \) converges because the sequence of its partial sums given by \( S_n \) converges (as seen in part (a)).

Sum of the series = \( \lim_{n \to \infty} S_n = \frac{2}{3} \).

(c) Find \( a_1 \) and \( a_2 \), i.e., find the first two terms of the series.

\[
a_1 = S_1 = \frac{2(1) + 5}{3(1) - 1} = \frac{7}{2}
\]

\[
a_2 = a_1 + a_2 \quad S_2 = \frac{2(2) + 5}{3(2) - 1} = \frac{9}{5}
\]

Also, \( S_2 = a_1 + a_2 \)

\[
\frac{9}{5} = \frac{7}{2} + a_2 \quad \text{So} \quad a_2 = \frac{9}{5} - \frac{7}{2} = \frac{-17}{10}
\]