QUIZ #8

Math 106-C (Salomone)

March 27, 2009

Show all your work!

Name: Solutions

Score (25 points possible):

*For each of the infinite series in # 1—5, **(a)** state your intuition on whether or not it converges conditionally, converges absolutely, or diverges, and **(b)** prove your assertion with your choice of convergence test(s).*

Problem 1. (5 points) $\displaystyle\sum_{n=0}^{\infty} \frac{79^n}{n!}$

(a) Dominated by factorial behavior. Expect asymptotic ratio $=0$ and hence convergence.

(b) $$r_n = \frac{a_n}{a_{n-1}} = \frac{\frac{79^n}{n!}}{\frac{79^{n-1}}{(n-1)!}} = \frac{79}{n} \xrightarrow{n\to\infty} 0.$$

Since $r = \lim_{n\to\infty} r_n = 0 < 1$, the ratio test implies this asymptotically geometric series converges. (Absolutely, since all terms are already positive.)

Problem 2. (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1/2)^n}{\sqrt{n+4}}$

(a) Dominated by geometric behavior, even after alternation is discarded. Expect convergence, since the ratios $(\frac{1}{2})$ are < 1.

(b) $$\lim |a_n| = \lim \left|\frac{(-1)^n}{2^n\sqrt{n+4}}\right| = \lim \frac{1}{2^n\sqrt{n+4}} \sim \frac{1}{\infty} = 0$$

Since $|a_n| \to 0$ we know $a_n \to 0$ and this converges by the alternating series test!

Moreover, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n+4}}$ has ratios

$$r_n = \frac{2^{n-1}\sqrt{n+3}}{2^n\sqrt{n+4}} = \frac{1}{2}\sqrt{\frac{n+3}{n+4}} \xrightarrow{n\to\infty} \frac{1}{2} < 1$$ giving absolute convergence by the ratio test.

Problem 3. (5 points) $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$

ⓐ Behavior looks like competing factorials. As $n \to \infty$ this approximately looks like the geometric series $\sum (1/3)^n$ which converges.

ⓑ Use the root characterization for asymptotic ratios:

$$R_n = \sqrt[n]{a_n} = \sqrt[n]{\left(\frac{n}{3n+1}\right)^n} = \frac{n}{3n+1} \xrightarrow{n\to\infty} \frac{1}{3} < 1,$$

giving absolute convergence by the ratio test.

Problem 4. (5 points) $\sum_{n=1}^{\infty}\frac{1}{16+n^2}$

ⓐ Were it not for the 16, would look like $\sum \frac{1}{n^2}$ which converges as a "fast polynomial" series.

ⓑ The integral test casts this series as a left sum for the integral $\int_0^\infty \frac{1}{16+x^2}\,dx$. Since $\frac{1}{16+x^2}$ is decreasing on $[0,\infty)$, the integral exceeds its left sum.

But $\int_0^\infty \frac{1}{16+x^2}dx = \frac{1}{4}\arctan\frac{x}{4}\Big|_0^\infty = \frac{\pi}{8}$ so in particular the integral is finite.

Thus $\sum_{n=1}^{\infty}\frac{1}{16+n^2}$ converges absolutely (to a value no bigger than $\frac{\pi}{8}$.)

Problem 5. (5 points) $\frac{2}{2}-\frac{4}{4}+\frac{8}{6}-\frac{16}{8}+\frac{32}{10}-\frac{64}{12}+\cdots$

$$= \sum_{n=1}^{\infty}\frac{2^n}{2n} = \sum_{n=1}^{\infty}\frac{2^{n-1}}{n}$$

ⓐ It appears the numerator, being geometric, will set the pace over the polynomial denominator. But the numerator $\to\infty$, so the terms $\to\infty$. Diverges.

ⓑ With l'Hôpital:

$$\lim_{n\to\infty}\frac{2^{n-1}}{n} \underset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty}\frac{(\ln 2)2^{n-1}}{1} = \frac{\infty}{1} = \infty$$

with asymptotic ratios:

$$r_n = \frac{2^{n-1}/n}{2^{n-2}/n-1} = \frac{2}{1}\,\frac{n-1}{n} \longrightarrow 2 > 1$$

Diverges by ratio test.