MATH 205 EXAM 2 MARCH 23, 2007

NAME: KEY

YOUR GRADE IS BASED ON CORRECTNESS, COMPLETENESS, AND CLARITY ON EACH EXERCISE. YOU MAY USE A CALCULATOR, BUT NO NOTES, BOOKS, OR OTHER STUDENTS. GOOD LUCK!

1.) (15 pts.) Use the Invertible Matrix Theorem to answer the following questions. In each of them, assume the matrix A is $n \times n$. Also, for each of them, fully explain your reasoning.

a.) (5 pts.) If there is an $n \times n$ matrix D for which $AD = I$, is there also an $n \times n$ matrix C for which $CA = I$?

Yes: these are parts (k) and (j) of the IMT. They are equivalent.

b.) (5 pts.) If A^T is not invertible, is A invertible?

No. Parts (a) and (l.) of the IMT are "A is invertible" and "A^T is invertible". Either <u>both</u> are true, or <u>both</u> or false.

c.) (5 pts.) If there is a vector $\mathbf{b}$ in $\mathbb{R}^n$ for which the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, can the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ be one-to-one?

The "if" part is equivalent to saying that part (g) of the IMT is <u>false</u>, hence all parts of the IMT is false. Then by part (f), the linear transformation $\vec{x} \mapsto A\vec{x}$ can <u>not</u> be 1-1.

2.) (15 pts.) Given

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ -1 & -3 & 2 & 4 & -8 \\ -2 & -1 & -6 & -7 & 9 \\ 5 & 6 & 8 & 7 & -5 \end{bmatrix},$$

compute a basis for Col A and a basis for Nul A. What is the dimension of Col A? What is the dimension of Nul A? What is rank A?

(Calculator):
$$A \sim \begin{bmatrix} \boxed{1} & 0 & 4 & 5 & -7 \\ 0 & \boxed{1} & -2 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So a $\boxed{\text{basis for Col } A}$ is $\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ 6 \end{bmatrix} \right\}$

and $\boxed{\dim \text{Col } A = 2}$

(and $\boxed{\text{rank } A = 2}$)

Solving $A\vec{x} = \vec{0}$, $\vec{x} = x_3 \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 7 \\ -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

So a $\boxed{\text{basis for Nul } A}$ is $\left\{ \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

and $\boxed{\dim \text{Nul } A = 3}$

3.) (15 pts.) In each of the following, assume the matrix A is 3×3. Also, for each of them, fully explain your reasoning.

a.) (5 pts.) Why is $\det A^T A \geq 0$?

Since $\det(A^T) = \det A$ and $\det(AB) = (\det A)(\det B)$

(both are theorems in Chapter 3), we know

$$\det A^T A = (\det A^T)(\det A) = (\det A)(\det A) = (\det A)^2$$

and any number squared is ≥ 0.

b.) (5 pts.) Is $\det(4A) = 4 \det A$?

No: $\det(4A) = 4^3 \det A$

(we can factor out a 4 from each of the 3 rows.)

c.) (5 pts.) If two rows of A are the same, then what is $\det A$?

In this case, $A \sim \begin{bmatrix} - & - & - \\ - & - & - \\ 0 & 0 & 0 \end{bmatrix}$

and we can show this with a single row operation that does not change the determinant, so

$$\det A = \det \begin{bmatrix} - & - & - \\ - & - & - \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

4.) (15 pts.) Compute all eigenvalues of the matrix A below. Then, for each eigenvalue, find a basis for its corresponding eigenspace.

$$A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) - 6$$

$$= \lambda^2 - \lambda - 12 = (\lambda - 4)(\lambda + 3) = 0 \text{ when } \lambda = -3, 4$$

So the $\boxed{\text{eigenvalues are } -3 \text{ and } 4.}$

$\lambda = -3$: $A - \lambda I = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{bmatrix}$

Solution to $(A - \lambda I)\vec{x} = \vec{0}$ are $\vec{x} = x_2 \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$.

Letting $x_2 = 3$, a basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$.

$\lambda = 4$: $A - \lambda I = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

Solution to $(A - \lambda I)\vec{x} = \vec{0}$ are $\vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Letting $x_2 = 1$, a basis for the eigenspace is $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

5.) (15 pts.) In each of the following, assume the matrix A is $n \times n$. Also, for each of them, fully explain your reasoning.

a.) (5 pts.) If A is diagonalizable, must A have n distinct eigenvalues?

No: it is the other way around.

(However, if A has an eigenvalue of multiplicity > 1, then the corresponding eigenspace should have the same dimension, in order for A to be diagonalizable.)

b.) (5 pts.) An eigenspace of A is the null space of which matrix?

$\boxed{A - \lambda I}$

That is, we are finding all $\vec{x}$ for which

$$(A - \lambda I)\vec{x} = \vec{0}.$$

c.) (5 pts.) Construct a nonzero 2×2 matrix that is diagonalizable but not invertible. (*Hint:* what must be true of the eigenvalues in such a matrix?)

Hint: one eigenvalue should be 0.

So, for example, $\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$ is diagonalizable

(in fact, it is already in diagonal form! We can let $P = I$.)

but not invertible.

6.) (15 pts.) Consider the set of vectors

$$\{\mathbf{x}, \mathbf{y}, \mathbf{z}\} = \left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \right\}.$$

a.) (5 pts.) Show that $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is an orthogonal set.

$$\left.\begin{aligned} \vec{x}\cdot\vec{y} &= -\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}-\tfrac{1}{4}=0 \\ \vec{x}\cdot\vec{z} &= \tfrac{1}{4}-\tfrac{1}{4}+\tfrac{1}{4}-\tfrac{1}{4}=0 \\ \vec{y}\cdot\vec{z} &= -\tfrac{1}{4}-\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}=0 \end{aligned}\right\}$$

All pairs have dot product 0, so all pairs are orthogonal, ∴ the set is orthogonal.

b.) (5 pts.) Show that $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is an orthonormal set.

By part (a) all pairs are orthogonal.

$$\text{Also, } \|\vec{x}\| = \|\vec{y}\| = \|\vec{z}\| = \sqrt{\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}} = \sqrt{1} = 1$$

So each is a unit vector, and the set is orthonormal.

c.) (5 pts.) Compute the projection of $\mathbf{y}$ on the vector $\mathbf{u} = \begin{bmatrix} -2 \\ 6 \\ 4 \\ 2 \end{bmatrix}$.

$$\text{proj}_{\vec{u}}\vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{1+3+2-1}{4+36+16+4}\vec{u} = \frac{5}{60}\vec{u} = \frac{1}{12}\vec{u} = \frac{1}{12}\begin{bmatrix} -2 \\ 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/6 \\ 1/2 \\ 1/3 \\ 1/6 \end{bmatrix}$$

7.) (10 pts.) A certain copy machine is always either working or broken (not working). If it is working today, there is a 70% chance that it will be working tomorrow. If it is broken today, there is a 50% chance that it will be broken tomorrow.

a.) (5 pts.) Assuming the copy machine is working today, what is the probability that it is working in two days?

$$\begin{array}{cc|c} \multicolumn{2}{c}{\text{Today}} & \\ \text{Working} & \text{Broken} & \\ \end{array}$$

$$\begin{array}{c} \\ \end{array}\begin{bmatrix} .7 & .5 \\ .3 & .5 \end{bmatrix} \begin{array}{l} \text{Working} \\ \text{Broken} \end{array}\Big\} \text{Tomorrow}$$

$$\vec{X}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Meaning: The copier is working today.

In two days:

$$\begin{bmatrix} .7 & .5 \\ .3 & .5 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .64 & .6 \\ .36 & .4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .64 \\ .36 \end{bmatrix}$$

$\therefore$ $\boxed{64\%}$ chance it will be working in two days

b.) (5 pts.) What is the long-term probability that the copy machine will be working on any given day? Use Markov chain techniques.

$$P - I = \begin{bmatrix} -.3 & .5 \\ .3 & -.5 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{5}{3} \\ 0 & 0 \end{bmatrix}$$

Long-term: $P\vec{q} = \vec{q}$. Solution for $\vec{q}$ is

$$\vec{q} = x_2 \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}.$$ First let $x_2 = 3$: $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Convert to a probability vector: $\frac{1}{8}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} .625 \\ .375 \end{bmatrix}$

Then in the long-term, there is a $\boxed{62.5\%}$ chance of the copy machine 7 working on any given day.