

NAME: KEY

YOUR GRADE IS BASED ON CORRECTNESS, COMPLETENESS, AND CLARITY ON EACH EXERCISE. YOU MAY USE A CALCULATOR, BUT NO NOTES, BOOKS, OR OTHER STUDENTS. GOOD LUCK!

1.) (15 pts.) Use the Invertible Matrix Theorem to answer the following questions. In each of them, assume the matrix A is $n \times n$. Also, for each of them, fully explain your reasoning.

a.) (5 pts.) If there is an $n \times n$ matrix D for which $AD = I$, is there also an $n \times n$ matrix C for which $CA = I$?

Yes: these are parts (k) and (j) of the IMT. They are equivalent.

b.) (5 pts.) If A^T is not invertible, is A invertible?

No. parts (a) and (d.) of the IMT are "A is invertible" and " A^T is invertible". Either both are true, or both are false.

c.) (5 pts.) If there is a vector \mathbf{b} in \mathbb{R}^n for which the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, can the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ be one-to-one?

The "if" part is equivalent to saying that part (g) of the IMT is false, hence all parts of the IMT is false. Then by part (f), the linear transformation $\vec{x} \mapsto A\vec{x}$ can not be 1-1.

2.) (15 pts.) Given

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ -1 & -3 & 2 & 4 & -8 \\ -2 & -1 & -6 & -7 & 9 \\ 5 & 6 & 8 & 7 & -5 \end{bmatrix},$$

compute a basis for Col A and a basis for Nul A . What is the dimension of Col A ? What is the dimension of Nul A ? What is rank A ?

(Calculator): $A \sim \begin{bmatrix} \boxed{1} & 0 & 4 & 5 & -7 \\ 0 & \boxed{1} & -2 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

So a basis for Col A is $\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ 6 \end{bmatrix} \right\}$
and $\dim \text{Col } A = 2$
(and $\text{rank } A = 2$)

Solving $A\vec{x} = \vec{0}$, $\vec{x} = x_3 \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 7 \\ -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

So a basis for Nul A is $\left\{ \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$
and $\dim \text{Nul } A = 3$

3.) (15 pts.) In each of the following, assume the matrix A is 3×3 . Also, for each of them, fully explain your reasoning.

a.) (5 pts.) Why is $\det A^T A \geq 0$?

Since $\det(A^T) = \det A$ and $\det(AB) = (\det A)(\det B)$
(both are theorems in Chapter 3), we know

$$\det A^T A = (\det A^T)(\det A) = (\det A)(\det A) = (\det A)^2$$

and any number squared is ≥ 0 .

b.) (5 pts.) Is $\det(4A) = 4 \det A$?

No: $\det(4A) = 4^3 \det A$

(We can factor out a 4 from each of the
3 rows.)

c.) (5 pts.) If two rows of A are the same, then what is $\det A$?

In this case, $A \sim \begin{bmatrix} - & - & - \\ - & - & - \\ 0 & 0 & 0 \end{bmatrix}$

and we can show this with a single row operation
that does not change the determinant, so

$$\det A = \det \begin{bmatrix} - & - & - \\ - & - & - \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

4.) (15 pts.) Compute all eigenvalues of the matrix A below. Then, for each eigenvalue, find a basis for its corresponding eigenspace.

$$A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3 - \lambda & 2 \\ 3 & -2 - \lambda \end{vmatrix} = (3 - \lambda)(-2 - \lambda) - 6$$

$$= \lambda^2 - \lambda - 12 = (\lambda - 4)(\lambda + 3) = 0 \text{ when } \lambda = -3, 4$$

So the eigenvalues are -3 and 4 .

$$\lambda = -3: A - \lambda I = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{bmatrix}$$

Solution to $(A - \lambda I)\vec{x} = \vec{0}$ are $\vec{x} = x_2 \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$.

Letting $x_2 = 3$, a basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$.

$$\lambda = 4: A - \lambda I = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Solution to $(A - \lambda I)\vec{x} = \vec{0}$ are $\vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Letting $x_2 = 1$, a basis for the eigenspace is $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

5.) (15 pts.) In each of the following, assume the matrix A is $n \times n$. Also, for each of them, fully explain your reasoning.

a.) (5 pts.) If A is diagonalizable, must A have n distinct eigenvalues?

No: it is the other way around.

(However, if A has an eigenvalue of multiplicity > 1 , then the corresponding eigenspace should have the same dimension, in order for A to be diagonalizable.)

b.) (5 pts.) An eigenspace of A is the null space of which matrix?

$$\boxed{A - \lambda I}$$

That is, we are finding all \vec{x} for which

$$(A - \lambda I) \vec{x} = \vec{0}.$$

c.) (5 pts.) Construct a nonzero 2×2 matrix that is diagonalizable but not invertible.
(Hint: what must be true of the eigenvalues in such a matrix?)

Hint: one eigenvalue should be 0.

So, for example, $\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$ is diagonalizable

(in fact, it is already in diagonal form!

We can let $P = I$.)

but not invertible.

6.) (15 pts.) Consider the set of vectors

$$\{x, y, z\} = \left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \right\}$$

a.) (5 pts.) Show that $\{x, y, z\}$ is an orthogonal set.

$$\left. \begin{aligned} \vec{x} \cdot \vec{y} &= -\frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{4} = 0 \\ \vec{x} \cdot \vec{z} &= \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} = 0 \\ \vec{y} \cdot \vec{z} &= -\frac{1}{4} - \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 0 \end{aligned} \right\} \begin{aligned} &\text{All pairs have dot product } 0, \\ &\text{so all pairs are orthogonal,} \\ &\therefore \text{the set is orthogonal.} \end{aligned}$$

b.) (5 pts.) Show that $\{x, y, z\}$ is an orthonormal set.

By part (a) all pairs are orthogonal.

$$\begin{aligned} \text{Also, } \|\vec{x}\| &= \|\vec{y}\| = \|\vec{z}\| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

So each is a unit vector, and the set is orthonormal.

c.) (5 pts.) Compute the projection of y on the vector $u = \begin{bmatrix} -2 \\ 6 \\ 4 \\ 2 \end{bmatrix}$.

$$\begin{aligned} \text{proj}_{\vec{u}} \vec{y} &= \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} \\ &= \frac{1+3+2-1}{4+36+16+4} \vec{u} \\ &= \frac{5}{60} \vec{u} \\ &= \frac{1}{12} \vec{u} \end{aligned} \quad \begin{aligned} &= \frac{1}{12} \begin{bmatrix} -2 \\ 6 \\ 4 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -1/6 \\ 1/2 \\ 1/3 \\ 1/6 \end{bmatrix} \end{aligned}$$

7.) (10 pts.) A certain copy machine is always either working or broken (not working). If it is working today, there is a 70% chance that it will be working tomorrow. If it is broken today, there is a 50% chance that it will be broken tomorrow.

a.) (5 pts.) Assuming the copy machine is working today, what is the probability that it is working in two days?

<u>Today</u>		$\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	Meaning: The copier is working today.
Working	Broken		

$\begin{bmatrix} .7 & .5 \\ .3 & .5 \end{bmatrix}$	Working Broken	Tomorrow
--	-------------------	----------

In two days:

$$\begin{bmatrix} .7 & .5 \\ .3 & .5 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .64 & .6 \\ .36 & .4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .64 \\ .36 \end{bmatrix} \therefore \boxed{64\%}$$

chance it will be working in two days

b.) (5 pts.) What is the long-term probability that the copy machine will be working on any given day? Use Markov chain techniques.

$$P - I = \begin{bmatrix} -.3 & .5 \\ .3 & -.5 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{5}{3} \\ 0 & 0 \end{bmatrix}$$

Long-term: $P\vec{q} = \vec{q}$. Solution for \vec{q} is

$$\vec{q} = x_2 \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}. \text{ First let } x_2 = 3: \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

Convert to a probability vector: $\frac{1}{8} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} .625 \\ .375 \end{bmatrix}$

Then in the long-term, there is a $\boxed{62.5\%}$ chance of the copy machine working on any given day.