Answer Key for Exam #2

1. Use elimination on an augmented matrix:

$$\begin{pmatrix} 1 & 3 & 1 & 2 & & 2 \\ 2 & 3 & 5 & 1 & & 4 \\ 3 & 4 & 1 & 8 & & 20 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 1 & 2 & & 2 \\ 0 & -3 & 3 & -3 & & 0 \\ 0 & -5 & -2 & 2 & & 14 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 4 & -1 & & 2 \\ 0 & 1 & -1 & 1 & & 0 \\ 0 & 0 & -7 & 7 & & 14 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 3 & & 10 \\ 0 & 1 & 0 & 0 & & -2 \\ 0 & 0 & 1 & -1 & & -2 \end{pmatrix}$$

The fourth column has no pivot, so x_4 is a free variable. The corresponding system is

$$x_1 + 3x_4 = 10, \qquad x_2 = -2, \qquad x_3 - x_4 = -2$$

which we solve for the pivot variables:

$$\begin{array}{rcll} x_1 & = & 10 & -3x_4 \\ x_2 & = & -2 & \\ x_3 & = & -2 & +x_4 \\ x_4 & = & & x_4 \end{array}$$

Therefore

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \\ -2 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

2. We perform the eliminations

$$A = \begin{pmatrix} 1 & 1 & 3 & 2 & 3 \\ 2 & 3 & 7 & 3 & 4 \\ 1 & 2 & 4 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 3 & 2 & 3 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R$$

A basis for the row space is the pivot rows of R, or of A. A basis for the column space is the pivot columns of A (but not of R). A basis for the nullspace can be found as in problem 1 or by taking the negative of the upper right corner

$$\begin{pmatrix} 2 & 3 & 5 \\ 1 & -1 & -2 \end{pmatrix}$$

of R, putting a 3×3 identity matrix below it, and taking the three columns of that. So the only basis that requires more work is the left nullspace. To get it we transpose the pivot columns of A and eliminate:

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

Here we can solve the corresponding system, or throw away the 2×2 identity on the left, negate the rest, and put a 1×1 identity under it. We also have another basis for the column space in the rows of the last matrix above. In conclusion

A **row space basis** is $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ -2 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \\ 3 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ 7 \\ 3 \\ 4 \end{pmatrix}$

A **null space basis** is
$$\begin{pmatrix}-2\\-1\\1\\0\\0\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}-3\\1\\0\\1\\0\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}-5\\2\\0\\0\\1\end{pmatrix}$$

A **column space basis** is
$$\begin{pmatrix}1\\2\\1\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}1\\3\\2\end{pmatrix} \qquad\text{or}\qquad \begin{pmatrix}1\\0\\-1\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}0\\1\\1\end{pmatrix}$$

A **left null space basis** is
$$\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

The factored form of A that displays bases for all four is

$$A = \begin{pmatrix}1 & 0\\0 & 1\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\2 & 3\end{pmatrix}\begin{pmatrix}1 & 0 & 2 & 3 & 5\\0 & 1 & 1 & -1 & -2\end{pmatrix}$$

3. To see which vector to keep we start by computing all the dot products for the three vectors. If

$$\vec{v}_1 = \begin{pmatrix}3\\2\\1\\2\end{pmatrix} \quad\text{and}\quad \vec{v}_2 = \begin{pmatrix}1\\5\\-1\\1\end{pmatrix} \quad\text{and}\quad \vec{v}_3 = \begin{pmatrix}2\\3\\4\\1\end{pmatrix}$$

then

$$\begin{aligned} &\vec{v}_1\cdot\vec{v}_2 = 3+10-1+2 = 14, && \vec{v}_1\cdot\vec{v}_3 = 6+6+4+2 = 18, && \vec{v}_2\cdot\vec{v}_3 = 2+15-4+1 = 14,\\ &\vec{v}_1\cdot\vec{v}_1 = 9+4+1+4 = 18, && \vec{v}_2\cdot\vec{v}_2 = 1+25+1+1 = 28, && \vec{v}_3\cdot\vec{v}_3 = 4+9+16+1 = 30 \end{aligned}$$

Recall that the projection of $\vec{b}$ onto $\vec{a}$ is $\dfrac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\,\vec{a}$. If we take $\vec{a} = \vec{v}_2$ then the ratios will both be $\frac{14}{28} = \frac{1}{2}$, so $\vec{v}_2$ seems like a good one to keep. Then the projection of $\vec{v}_1$ onto $\vec{v}_2$ is

$$\frac{\vec{v}_1\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\,\vec{v}_2 = \frac{14}{28}\begin{pmatrix}1\\5\\-1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\5\\-1\\1\end{pmatrix},$$

and therefore

$$\vec{v}_1 = \begin{pmatrix}3\\2\\1\\2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\5\\-1\\1\end{pmatrix} + \vec{e},$$

where $\vec{e}$ is the error in the projection. We want to replace $\vec{v}_1$ by some multiple of $\vec{e}$. We have

$$2\vec{e} = \begin{pmatrix}6\\4\\2\\4\end{pmatrix} - \begin{pmatrix}1\\5\\-1\\1\end{pmatrix} = \begin{pmatrix}5\\-1\\3\\3\end{pmatrix} = \vec{w}_1,$$

so we throw away $\vec{v}_1$ and replace it by $\vec{w}_1$. Next we do the same thing with $\vec{v}_3$. The projection of $\vec{v}_3$ onto $\vec{v}_2$ is

$$\frac{\vec{v}_3\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\,\vec{v}_2 = \frac{14}{28}\begin{pmatrix}1\\5\\-1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\5\\-1\\1\end{pmatrix},$$

and therefore

$$\vec{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 5 \\ -1 \\ 1 \end{pmatrix} + \vec{e},$$

where $\vec{e}$ is the error in the projection, and again we want to replace $\vec{v}_3$ by some multiple of $\vec{e}$. We have

$$2\vec{e} = \begin{pmatrix} 4 \\ 6 \\ 8 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 9 \\ 1 \end{pmatrix} = \vec{w}_3,$$

so we throw away $\vec{v}_3$ and replace it by $\vec{w}_3$. If we rename $\vec{v}_2$ as $\vec{w}_2$, we now have

$$\vec{w}_1 = \begin{pmatrix} 5 \\ -1 \\ 3 \\ 3 \end{pmatrix} \quad \text{and} \quad \vec{w}_2 = \begin{pmatrix} 1 \\ 5 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{w}_3 = \begin{pmatrix} 3 \\ 1 \\ 9 \\ 1 \end{pmatrix},$$

where $\vec{w}_1 \perp \vec{w}_2$ and $\vec{w}_3 \perp \vec{w}_2$, but usually we would not have $\vec{w}_1 \perp \vec{w}_3$ at this point. (You might, if you're lucky; in fact, if we had decided to keep $\vec{v}_1$ instead of $\vec{v}_2$ then we *would* have been lucky at this stage.) In this case $\vec{w}_1 \cdot \vec{w}_3 = 15 - 1 + 27 + 3 = 44 \neq 0$, so they are not perpendicular. Since we also have $\vec{w}_1 \cdot \vec{w}_1 = 25 + 1 + 9 + 9 = 44$ and $\vec{w}_3 \cdot \vec{w}_3 = 9 + 1 + 81 + 1 = 92$, it is a good idea to keep $\vec{w}_1$ and change $\vec{w}_3$. The projection of $\vec{w}_3$ onto $\vec{w}_1$ is

$$\frac{\vec{w}_1 \cdot \vec{w}_3}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 = \frac{44}{44}\begin{pmatrix} 5 \\ -1 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \\ 3 \end{pmatrix},$$

and therefore

$$\vec{w}_3 = \begin{pmatrix} 3 \\ 1 \\ 9 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \\ 3 \end{pmatrix} + \vec{e} = \begin{pmatrix} 5 \\ -1 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \\ 6 \\ -2 \end{pmatrix}.$$

Replacing $\vec{w}_3$ by the simplest multiple of $\vec{e}$, we now have

$$\begin{pmatrix} 5 \\ -1 \\ 3 \\ 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 5 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 1 \\ 3 \\ -1 \end{pmatrix},$$

which are all perpendicular to each other, so we only have to fix the lengths. The dot product of the last vector with itself is 12, and we did the others earlier, so we finally get that an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ is

$$\frac{1}{\sqrt{44}}\begin{pmatrix} 5 \\ -1 \\ 3 \\ 3 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{28}}\begin{pmatrix} 1 \\ 5 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{12}}\begin{pmatrix} -1 \\ 1 \\ 3 \\ -1 \end{pmatrix}.$$

If we had kept $\vec{w}_3$ and changed $\vec{w}_1$ we would have instead

$$\frac{1}{\sqrt{92}}\begin{pmatrix} 3 \\ 1 \\ 9 \\ 1 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{28}}\begin{pmatrix} 1 \\ 5 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{3036}}\begin{pmatrix} 41 \\ -17 \\ -15 \\ 29 \end{pmatrix}.$$

If we had kept $\vec{v}_1$ initially and changed the others we would have wound up with the orthonormal basis

$$\frac{1}{\sqrt{18}}\begin{pmatrix}3\\2\\1\\2\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{1386}}\begin{pmatrix}12\\-31\\16\\5\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{12}}\begin{pmatrix}-1\\1\\3\\-1\end{pmatrix},$$

and if we had kept $\vec{v}_3$ at the beginning and changed the others we could have come out with

$$\frac{1}{\sqrt{30}}\begin{pmatrix}2\\3\\4\\1\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{1386}}\begin{pmatrix}12\\-31\\16\\5\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{180}}\begin{pmatrix}9\\1\\-7\\7\end{pmatrix}$$

or with

$$\frac{1}{\sqrt{30}}\begin{pmatrix}2\\3\\4\\1\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{3036}}\begin{pmatrix}41\\-17\\-15\\29\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{4830}}\begin{pmatrix}1\\54\\-43\\8\end{pmatrix}.$$

4. Let A be the matrix with $\vec{v}_1$ and $\vec{v}_2$ as columns. Then

$$A^TA = \begin{pmatrix}1 & 3 & 2 & 1 & 1\\1 & 1 & 2 & 3 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\3 & 1\\2 & 2\\1 & 3\\1 & 1\end{pmatrix} = \begin{pmatrix}16 & 12\\12 & 16\end{pmatrix},$$

and

$$\begin{pmatrix}16 & 12\\12 & 16\end{pmatrix}^{-1} = \frac{1}{16\cdot 16 - 12\cdot 12}\begin{pmatrix}16 & -12\\-12 & 16\end{pmatrix} = \frac{1}{28}\begin{pmatrix}4 & -3\\-3 & 4\end{pmatrix},$$

so

$$P = \frac{1}{28}\begin{pmatrix}1 & 1\\3 & 1\\2 & 2\\1 & 3\\1 & 1\end{pmatrix}\begin{pmatrix}4 & -3\\-3 & 4\end{pmatrix}\begin{pmatrix}1 & 3 & 2 & 1 & 1\\1 & 1 & 2 & 3 & 1\end{pmatrix} = \frac{1}{28}\begin{pmatrix}1 & 1\\9 & -5\\2 & 2\\-5 & 9\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 3 & 2 & 1 & 1\\1 & 1 & 2 & 3 & 1\end{pmatrix},$$

and so we find that the projection matrix P onto the subspace S is

$$P = \frac{1}{28}\begin{pmatrix}2 & 4 & 4 & 4 & 2\\4 & 22 & 8 & -6 & 4\\4 & 8 & 8 & 8 & 4\\4 & -6 & 8 & 22 & 4\\2 & 4 & 4 & 4 & 2\end{pmatrix} = \frac{1}{14}\begin{pmatrix}1 & 2 & 2 & 2 & 1\\2 & 11 & 4 & -3 & 2\\2 & 4 & 4 & 4 & 2\\2 & -3 & 4 & 11 & 2\\1 & 2 & 2 & 2 & 1\end{pmatrix}$$

We also have that

$$R = 2P - I = \frac{1}{7}\begin{pmatrix}1 & 2 & 2 & 2 & 1\\2 & 11 & 4 & -3 & 2\\2 & 4 & 4 & 4 & 2\\2 & -3 & 4 & 11 & 2\\1 & 2 & 2 & 2 & 1\end{pmatrix} - \frac{1}{7}\begin{pmatrix}7 & 0 & 0 & 0 & 0\\0 & 7 & 0 & 0 & 0\\0 & 0 & 7 & 0 & 0\\0 & 0 & 0 & 7 & 0\\0 & 0 & 0 & 0 & 7\end{pmatrix} = \frac{1}{7}\begin{pmatrix}-6 & 2 & 2 & 2 & 1\\2 & 4 & 4 & -3 & 2\\2 & 4 & -3 & 4 & 2\\2 & -3 & 4 & 4 & 2\\1 & 2 & 2 & 2 & -6\end{pmatrix}$$

is the reflection matrix through S. The projection of $\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 4 \\ 3 \\ 1 \end{pmatrix}$ onto S is

$$P\vec{v}_3 = \frac{1}{14}\begin{pmatrix} 1 & 2 & 2 & 2 & 1 \\ 2 & 11 & 4 & -3 & 2 \\ 2 & 4 & 4 & 4 & 2 \\ 2 & -3 & 4 & 11 & 2 \\ 1 & 2 & 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 4 \\ 3 \\ 1 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 14 \\ 0 \\ 28 \\ 56 \\ 14 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 1 \end{pmatrix},$$

and the reflection of $\vec{v}_3$ through S is

$$R\vec{v}_3 = \frac{1}{7}\begin{pmatrix} -6 & 2 & 2 & 2 & 1 \\ 2 & 4 & 4 & -3 & 2 \\ 2 & 4 & -3 & 4 & 2 \\ 2 & -3 & 4 & 4 & 2 \\ 1 & 2 & 2 & 2 & -6 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 4 \\ 3 \\ 1 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 7 \\ 7 \\ 0 \\ 35 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 5 \\ 1 \end{pmatrix}$$

The projection is the average of the reflection and $\vec{v}_3$ itself, and this could have been used to avoid one of the last two matrix multiplications.

5. If $P = \frac{1}{20}\begin{pmatrix} 1 & 3 & 3 & 1 \\ 3 & 11 & 9 & -3 \\ 3 & 9 & 9 & 3 \\ 1 & -3 & 3 & 19 \end{pmatrix}$ then P is symmetric and

$$P^2 = \frac{1}{400}\begin{pmatrix} 20 & 60 & 60 & 20 \\ 60 & 220 & 180 & -60 \\ 60 & 180 & 180 & 60 \\ 20 & -60 & 60 & 380 \end{pmatrix} = P.$$

Any matrix P which satisfies $P^2 = P = P^T$ is a projection matrix. The trace of P is $\frac{1}{20}(1 + 11 + 9 + 19) = 2$, so the subspace T that P projects onto is 2-dimensional. Therefore any two rows or columns of P will be a basis for it as long as they are not multiples of each other. We can use the same trick on $I - P$ to get a basis for $T^\perp$. Since only one of the rows of P is very nice, though, let's eliminate:

$$P = \frac{1}{20}\begin{pmatrix} 1 & 3 & 3 & 1 \\ 3 & 11 & 9 & -3 \\ 3 & 9 & 9 & 3 \\ 1 & -3 & 3 & 19 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 2 & 0 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 18 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 3 & 10 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

P projects onto its own column space, or its row space since P is symmetric. This gives us a nice basis for T, and a basis for $T^\perp$ comes from the null space of P:

A basis for T is $\begin{pmatrix} 1 \\ 0 \\ 3 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ -3 \end{pmatrix}$ and **A basis for $T^\perp$** is $\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -10 \\ 3 \\ 0 \\ 1 \end{pmatrix}$

6. We would like to find the line $y = mx + b$ which best fits the four points $(1, 2)$, $(2, 1)$, $(3, 3)$ and $(4, 2)$ in the sense of least squares. If the line fit exactly we would have

$$\begin{aligned} 2 &= m + b \\ 1 &= 2m + b \\ 3 &= 3m + b \\ 2 &= 4m + b \end{aligned} \quad \text{or} \quad \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix}$$

This equation has no solution, but we can find the least squares solution by multiplying by the transpose of the matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix}$$

which becomes

$$\begin{pmatrix} 30 & 10 \\ 10 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 21 \\ 8 \end{pmatrix}.$$

Solving this by elimination on an augmented matrix we get

$$\begin{pmatrix} 30 & 10 & & 21 \\ 10 & 4 & & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & -2 & & -3 \\ 5 & 2 & & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 2 & & 3 \\ 5 & 0 & & 1 \end{pmatrix}.$$

Therefore $2b = 3$ and $5m = 1$, so $m = \frac{1}{5}$ and $b = \frac{3}{2}$ and the best line is $y = \dfrac{x}{5} + \dfrac{3}{2}$.

7. If U is a 6-dimensional subspace of $\mathbb{R}^8$ then the projection matrix onto U is $P = A\left(A^T A\right)^{-1} A^T$, where A is any 8×6 matrix whose columns are a basis of A. Calculating P directly by hand would not be fun since we would have to invert the 6×6 matrix $A^T A$. According to the fundamental theorem of linear algebra, the left null space of A is an $8 - 6 = 2$ dimensional subspace of $\mathbb{R}^8$ and it is the orthogonal complement of the column space of A. In other words, $U^\perp$ is a 2-dimensional subspace of $\mathbb{R}^8$ and we know how to find a basis for it. Thus an indirect way to find P is as follows:

(i) find a basis $\{\vec{v}_1, \vec{v}_2\}$ for the left null space of A
(ii) let $\mathcal{A}$ be the matrix with $\vec{v}_1$ and $\vec{v}_2$ as columns, and calculate $\mathcal{P} = \mathcal{A}\left(\mathcal{A}^T \mathcal{A}\right)^{-1} \mathcal{A}^T$
(iii) $\mathcal{P}$ is the projection matrix onto $U^\perp$, and therefore $I - \mathcal{P}$ is the projection matrix onto U.

Step (i) takes some work, but in (ii) we only have to invert a 2×2 matrix.

Scores: For the 21 exams which I have at this writing, the median is 90 and the mean is about 88.7.

Score	Frequency	Score	Frequency	Score	Frequency
99	1	91	2	83	1
96	2	90	1	82	2
95	3	89	3	79	1
94	1	88	1	63	1
92	1	85	1	??	2