MATH 205 EXAM 2 MARCH 20, 2008

NAME: KEY

YOUR GRADE IS BASED ON CORRECTNESS, COMPLETENESS, AND CLARITY ON EACH EXERCISE. YOU MAY USE A CALCULATOR, BUT NO NOTES, BOOKS, OR OTHER STUDENTS. GOOD LUCK!

1.) (15 pts.)

a.) (10 pts.) Consider the subset of $\mathbb{P}_2$ of all polynomials of the form $\mathbf{p}(t) = a + bt^2$, where a is in $\mathbb{R}$ and b is in $\mathbb{R}$. Demonstrate that this subset is a subspace of $\mathbb{P}_2$.

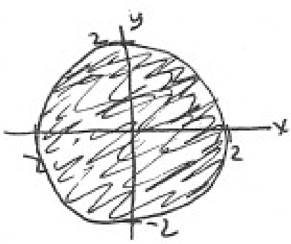
1) Let $\vec{p}(t) = a+bt^2$, with a and b in $\mathbb{R}$, and $\vec{q}(t) = c+dt^2$, with c and d in $\mathbb{R}$. Then $(\vec{p}+\vec{q})(t) = (a+bt^2)+(c+dt^2) = (a+c)+(b+d)t^2$, with a+c in $\mathbb{R}$ and b+d in $\mathbb{R}$. Therefore $(\vec{p}+\vec{q})(t)$ is in the set.

2) Let $\vec{p}(t) = a+bt^2$, with a and b in $\mathbb{R}$, and c be a real scalar. Then $(c\vec{p})(t) = c(a+bt^2) = ca+cbt^2$ with ca and cb in $\mathbb{R}$. Therefore $(c\vec{p})(t)$ is in the set.

3.) The zero vector is in the set. To see this, let $a=b=0$.

These are the three things we needed to verify to show the set is a subspace.

b.) (5 pts.) Let H be the set of points inside and on a circle of radius 2 that is centered at the origin of the xy-plane. That is, $H = \{(x, y) : x^2 + y^2 \leq 4\}$. Use an example (two vectors, or a vector and a scalar) to show that H is *not* a subspace of $\mathbb{R}^2$.



$\vec{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is in H. However, for the scalar 5, $5\vec{x} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$ is not in H.

This means H is not closed under scalar multiplication, which means H is not a subspace of $\mathbb{R}^2$.

2.) (15 pts.) Use the Invertible Matrix Theorem (IMT) to respond to the following questions. Be sure to state clearly which parts of the IMT you are using. In all cases, assume the matrix A is $n \times n$.

a.) (5 pts.) If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then does A have fewer than n pivot positions?

$A\vec{x} = \vec{0}$ has a nontrivial solution, meaning (d) of IMT is false. So: all parts of IMT are false.

In particular, (c) is false, so A does not have n pivot positions.

Therefore, yes, A has fewer than n pivot positions.

(An $n \times n$ matrix can never have more than n pivot positions.)

b.) (5 pts.) If 0 is *not* an eigenvalue of A, is A^T invertible?

0 is not an eigenvalue: part (s) is true.

So: all parts of IMT are true.

In particular: (l.) is true.

So A^T is an invertible matrix.

c.) (5 pts.) Suppose the columns of A form a basis for $\mathbb{R}^n$. What can you say about dim Col A? What can you say about dim Nul A?

Columns of A form a basis for $\mathbb{R}^n$: that is, part (m) is true.

Therefore all parts are true.

In particular, by part (o.), dim Col A = n.

Also, by part (r), dim Nul A = 0.

3.) (15 pts.)

a.) (5 pts.) Consider the set $\left\{ \begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix} : a,b,c,d \text{ real} \right\}$. Find a matrix A, having linearly independent columns, for which this set is Col A.

The set is spanned by $\vec{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$.

The matrix formed with these vectors has RREF $\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & -2 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

So columns 1, 2, 4 are linearly independent.

Then $A = \begin{bmatrix} 1 & -3 & 0 \\ 5 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix}$ has column space equal to the given set.

b.) (5 pts.) For the matrix $B = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, find a basis for Nul B.

$B \sim \begin{bmatrix} 1 & 0 & 6 & -8 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $B\vec{x} = \vec{0}$ has solutions

$\vec{x} = x_3 \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ so $\left\{ \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul B

c.) (5 pts.) For the matrix B above, what is dim Nul B? What is rank B?

dim Nul B = 3 . rank B = 2

$\downarrow$ $\downarrow$

free variables # pivot positions.

4.) (15 pts.)

a.) (7 pts.) Is $\lambda = 2$ an eigenvalue of $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$? Why or why not?

$\lambda = 2$ is an eigenvalue if $(A - 2I)\vec{x} = \vec{0}$ has nontrivial solutions.

Let's see:

$$A - 2I = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & .5 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

By IMT (c), $A-2I$ does <u>not</u> have n $(=3)$ pivot positions, hence by (d), $A-2I$ does <u>not</u> have only the trivial solution. So $A-2I$ <u>has</u> nontrivial solutions, and 2 is an eigenvalue of A.

b.) (8 pts.) Is $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ an eigenvector of $B = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$? Why or why not?

Does $B\vec{u} = \lambda \vec{u}$ for some scalar λ?

$B\vec{u} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\vec{u}$, so <u>yes</u>, $\vec{u}$ is an eigenvector of B (with eigenvalue -4).

5.) (15 pts.) Use any theorems or other results from this semester to respond to the following questions. Be sure to state clearly which theorems, parts of theorems, or other reasoning you are using.

a.) (5 pts.) Let A be an $n \times n$ matrix. If the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then is there a *unique* solution for each $\mathbf{b}$ in $\mathbb{R}^n$?

$A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$, therefore part (g) of IMT is <u>true</u>.

This means part (c) of IMT is <u>true</u>: A has n pivot positions.

Since A is $n \times n$, then, A has <u>no free variables</u>.

Therefore there <u>is</u> a unique solution for each $\vec{b}$ in $\mathbb{R}^n$.

b.) (5 pts.) If $n \times n$ matrices E and F have the property that $EF = I$, then must $FE = I$?

If $EF = I$, then by an unnamed theorem in Section 2.3, E and F are each invertible, with $F = E^{-1}$ and $E = F^{-1}$.

Then $FE = I$, as well, by definition of inverses.

c.) (5 pts.) Let B be an $m \times n$ matrix, where m and n are not necessarily equal. Is it still true that dim Row B = dim Col B?

Yes. Both dim Row B and dim Col B are equal to the number of pivot positions in B.

(We saw this in our work from Section 4.6)

6.) (15 pts.) Diagonalize the matrix $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$. *Hint: be clever when computing the eigenvalues.*

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 0 & -2 \\ 2 & 5-\lambda & 4 \\ 0 & 0 & 5-\lambda \end{vmatrix}$$

$$= (5-\lambda) \begin{vmatrix} 4-\lambda & 0 \\ 2 & 5-\lambda \end{vmatrix}$$

(expanding along bottom row)

$$= (5-\lambda)(4-\lambda)(5-\lambda)$$

$$= 0 \text{ for } \lambda = 4, 5$$

$\lambda = 4$: $A - 4I =$

$$\begin{bmatrix} 0 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & .5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution to $(A-4I)\vec{x} = \vec{0}$ are $\vec{x} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$.

Basis for eigenspace: $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

$\lambda = 5$: $A - 5I =$

$$\begin{bmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

($\lambda = 5$, continued)

~~Basis for eigenspace~~

Solutions to $(A-5I)\vec{x} = \vec{0}$ are

$$\vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Basis for eigenspace: $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Diagonalize: $A = PDP^{-1}$ where

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

7.) (10 pts.) Let $A = \begin{bmatrix} -2 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix}$. Complete the following tasks *by hand*, showing appropriate work. Use the back of this page if you need more room.

a.) (5 pts.) Compute AB for $B = \begin{bmatrix} 4 & -3 \\ 5 & -2 \\ 2 & -1 \end{bmatrix}$.

$$AB = \begin{bmatrix} -2 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 5 & -2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -8+15+2 & 6-6-1 \\ 0+5+0 & 0-2+0 \\ 4-10+0 & -3+4+0 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -1 \\ 5 & -2 \\ -6 & 1 \end{bmatrix}$$

b.) (5 pts.) Use the algorithm discussed in Chapter 2 to compute A^{-1}.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} -2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -2 & 3 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right]$$

$$= [I \mid A^{-1}]$$

So $A^{-1} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$.