Math 205A Winter 10
Test 2 (50 points)

Name: Solutions.

- Check that you have 6 questions on three pages.
- Show all your work to receive full credit for a problem.

1. (6 points) Let $C = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & -5 & 2 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix}$. Find all the eigenvalues of C.

$$C - \lambda I = \begin{bmatrix} 3-\lambda & 0 & 0 & 0 \\ 4 & 1-\lambda & -5 & 2 \\ 0 & 0 & 2-\lambda & 0 \\ 1 & 0 & 1 & 2-\lambda \end{bmatrix}$$

$$\det(C-\lambda I) = \det\left(\begin{bmatrix} 3-\lambda & 0 & 0 & 0 \\ 4 & 1-\lambda & -5 & 2 \\ 0 & 0 & 2-\lambda & 0 \\ 1 & 0 & 1 & 2-\lambda \end{bmatrix}\right) \text{ (expand across first row.)}$$

$$= (3-\lambda)\det\left(\begin{bmatrix} 1-\lambda & -5 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{bmatrix}\right) \text{ (expand across first column.)}$$

$$= (3-\lambda)(1-\lambda)\det\left(\begin{bmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix}\right).$$

$$= (3-\lambda)(1-\lambda)[(2-\lambda)(2-\lambda) - 0].$$

So $\det(C-\lambda I) = 0$ gives

$$(3-\lambda)(1-\lambda)(2-\lambda)^2 = 0$$

Hence, $\lambda = 3, 1$ or 2.

So eigenvalues of C are $3, 1$ and 2.

2. (6 points) Suppose 0 is an eigenvalue of a 6×6 matrix A and $\vec{u}$ is an eigenvector of A corresponding to the eigenvalue 0.

(a) Is A invertible? Explain.

Since 0 is an eigenvalue of A,
$\det(A - 0 \cdot I) = 0$ i.e $\det A = 0$
Hence A is not invertible.

(b) Is $\vec{u}$ an eigenvector of A^2? If so, find the corresponding eigenvalue. If not, explain why not.

Since $\vec{u}$ is an eigenvector of A corresponding to 0,
$A\vec{u} = 0 \cdot \vec{u} = \vec{0}$.
So $A^2\vec{u} = A(A\vec{u}) = A(\vec{0}) = \vec{0}$.
Thus $A^2\vec{u} = 0 \cdot \vec{u}$
Hence $\vec{u}$ is an eigenvector of A^2 and the corresponding eigenvalue is 0.

3. (12 points) Let $B = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ -3 & 6 & 2 \end{bmatrix}$. The eigenvalues of B are 2 and 4.

(a) Find a basis for the eigenspace corresponding to each eigenvalue of B.

For eigenvalue 2, we solve the equation $(B-2I)\vec{x}=\vec{0}$

$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ -3 & 6 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\vec{x} = x_2\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Basis for eigenspace of 2 $= \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\left(\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a linearly ind. set} \right)$.

For eigenvalue 4, we solve the equation $(B-4I)\vec{x}=\vec{0}$

$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -3 & 6 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. $\vec{x} = x_3\begin{bmatrix} 0 \\ 1/3 \\ 1 \end{bmatrix}$

So basis for eigenspace of 4 is $\left\{ \begin{bmatrix} 0 \\ 1/3 \\ 1 \end{bmatrix} \right\}$.

(We can also choose $\left\{ \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$ as basis.)

(b) Is $\vec{x} = \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}$ an eigenvector of B? If so, find the coordinates of $\vec{x}$ with respect to the basis (you found in part (a)) of the eigenspace to which it belongs. If not, explain why not.

$B\vec{x} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ -3 & 6 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 14 \end{bmatrix} = 2\begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix} = 2\vec{x}$. So $\vec{x}$ is an eigenvector of B and the corresponding eigenvalue is 2.

So we use the basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ to find $[\vec{x}]_{\mathcal{B}}$.

$\left[\begin{array}{cc|c} 2 & 0 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{array}\right]$ So $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ (coordinates of $\vec{x}$ w.r.t the basis of eigenspace for 2).

(c) Is B diagonalizable? If so, find the matrices P and D so that $B = PDP^{-1}$. If not, explain why not.

B is diagonalizable since B has three linearly independent eigenvectors (the bases for the two eigenspaces combined give three linearly independent eigenvectors.)

$P = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$$\begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array}$$

4. (10 points) Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \right\}$.

(a) Find a basis and the dimension of W.

$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 5 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Every column does not have a pivot. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is not a linearly independent set. In fact, $\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$. and $\{\vec{v}_1, \vec{v}_2\}$ is a lin. ind. set.

Also $W = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. ~~and~~ So $\{\vec{v}_1, \vec{v}_2\}$ is a basis for W.

dim $W = 2$.

(b) Find a basis and the dimension of $W^{\perp}$ (the orthogonal complement of W).

Let $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is ~~be~~ in $W^{\perp}$ if and only if ~~Then~~ $\vec{v}_1 \cdot \vec{u} = 0$ and $\vec{v}_2 \cdot \vec{u} = 0$ (since $\{\vec{v}_1, \vec{v}_2\}$ is a basis for W).

So we get $a + 2b + 5c = 0$

and $a + b + 3c = 0$.

$$\begin{bmatrix} 1 & 2 & 5 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

So $\vec{u} = c \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$. Hence $W^{\perp} = \text{Span}\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Basis for $W^{\perp} = \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$. dim $W^{\perp} = 1$.

(c) Give a geometric description of W and $W^{\perp}$.

W is a two-dimensional subspace of $\mathbb{R}^3$.

So W is a plane in $\mathbb{R}^3$ ~~passing~~ containing the origin and the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$W^{\perp}$ is a ~~t~~ one-dimensional subspace of $\mathbb{R}^3$.

So $W^{\perp}$ is a line in $\mathbb{R}^3$ passing through the origin and the vector $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$.

5. (8 points) Let A be a 2×2 matrix and B be a $2 \times \overset{2}{\cancel{3}}$ matrix, with $\det A = -4$ and $\det B = 7$.

(a) Find $\det 3B$.

B has 2 rows.
So $\det 3B = 3^2 \cdot \det B = 9 * 7 = 63$

(b) Find $\det A^3$. Is A^3 an invertible matrix? Explain.

$\det A^3 = (\det A)(\det A)(\det A) = (-4)^3 = -64$.

Since $\det A^3 \neq 0$,
A^3 is an invertible matrix.

(c) What is the largest possible rank of B? What is the smallest possible dimension of Nul B? Provide explanations for your answers.

$\det B = 7$. So B is an invertible matrix.
$B \sim I_2$. So rank of $B = 2$. and $\dim \operatorname{Nul} B = 0$.

(number of pivot columns)

(number of free variables in $B\vec{x} = \vec{0}$.)

6. (8 points) Let $\vec{v} = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 0 \end{bmatrix}$ and let $\vec{w} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix}$.

(a) Find the distance between $\vec{v}$ and $\vec{w}$.

$$\|\vec{v}-\vec{w}\| = \left\| \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix} \right\| = \sqrt{1+1+9+0} = \sqrt{11}.$$

(b) Compute $(\vec{v} \cdot \vec{w})\,\vec{v}$.

$$\vec{v}\cdot\vec{w} = 12+0-2+0 = 10.$$

$$10\vec{v} = \begin{bmatrix} 40 \\ 10 \\ 20 \\ 0 \end{bmatrix}.$$

(c) Find a vector of length 5 in the direction of $\vec{w}$.

$$\|\vec{w}\| = \sqrt{9+0+1+0} = \sqrt{10}.$$

So $\frac{1}{\sqrt{10}}\vec{w}$ is a vector of length one in the direction of $\vec{w}$.

Hence $5 \cdot \frac{1}{\sqrt{10}}\vec{w}$ is a vector of length 5 in the direction of $\vec{w}$.

$$5 \cdot \frac{1}{\sqrt{10}}\vec{w} = \begin{bmatrix} 15/\sqrt{10} \\ 0 \\ -5/\sqrt{10} \\ 0 \end{bmatrix}.$$