Math 205 (Winter 2016)

Test 2 (50 points)

Name: Solutions.

- Check that you have eight questions on three pages.
- Show all your work to receive full credit for a problem. Points will be taken off if you do not show how you arrived at your answer, even if the answer is correct.
- Please keep your explanations brief; be clear and to the point. Points will be taken off for incorrect or irrelevant statements.

1. (6 points) Let A and B be 6×6 matrices, with $\det A = -10$ and $\det B = 5$. Use properties of determinants to compute:

(a) $\det 3A$

$\det 3A = 3^6 \det A = 729(-10) = -7290.$

(b) $\det(A^T B^{-1})$

$$\det(A^T B^{-1}) = \det(A^T) \cdot \det(B^{-1})$$
$$= (\det A) \frac{1}{\det B}$$
$$= -10 \cdot \frac{1}{5}$$
$$= -2.$$

2. (7 points) Let $A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$. This matrix A has only one eigenvalue which is 5.

(a) Find a basis for the eigenspace corresponding to the eigenvalue 5.

To find a basis for the eigenspace, we solve the eqn.
$(A-5I)\bar{x} = \bar{0}$ ie $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\bar{x} = \bar{0}$.

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow$ This is in RREF.

General soln. is x_1 free, $x_2 = 0$.
Solution in parametric vector form is $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Thus basis for eigenspace corresponding to eigenvalue 5 is $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

(b) Is the matrix A diagonalizable? Explain.

The dimension of the eigenspace corresponding to eigenvalue 5 is 1 since a basis for this eigenspace is $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ as shown in part (a).
Since 5 is the only eigenvalue of A which is a 2x2 matrix, sum of the dimensions of all eigenspaces of $A = 1 \neq 2$.
Hence A is not diagonalizable.

3. (9 points) Let $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$.

(a) Find a unit vector in the direction of $\vec{v}$.

Unit vector in the direction of $\bar{v} = \frac{1}{\|\bar{v}\|} \cdot \bar{v}$.

$\|\bar{v}\| = \sqrt{\bar{v}\cdot\bar{v}} = \sqrt{1+9+0} = \sqrt{10}$.

$\frac{1}{\|\bar{v}\|} \cdot \bar{v} = \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \\ 0 \end{bmatrix}$.

(b) Find the distance between $\vec{v}$ and $\vec{y}$.

Distance $= \|\bar{v} - \bar{y}\| = \left\| \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix} \right\|$

$= \sqrt{9+9+25}$

$= \sqrt{43}$.

(c) Let $L = \text{Span}\{\vec{v}\}$. Compute the orthogonal projection of $\vec{y}$ onto L.

Orthogonal projection of $\bar{y}$ onto L

$= \hat{y} = \left(\frac{\bar{y}\cdot\bar{v}}{\bar{v}\cdot\bar{v}}\right)\bar{v} = \frac{-2+0+0}{1+9+0}\bar{v}$

$= -\frac{2}{10}\bar{v} = \frac{-1}{5}\bar{v}$

$= \begin{bmatrix} 1/5 \\ -3/5 \\ 0 \end{bmatrix}$.

4. (4 points) Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^4$. Is the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ a basis for $\mathbb{R}^4$? Explain.

An orthogonal set of non-zero vectors is linearly independent and so the set $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ is linearly independent.

Let $A = [\bar{v}_1\ \bar{v}_2\ \bar{v}_3\ \bar{v}_4]$. Since the columns of A are linearly independent, A has a pivot position in every column because there are no free variables in the eqn. $A\bar{x} = \bar{0}$. The matrix A is a 4×4 matrix and thus there is a pivot position in every row of A. Hence columns of A, ie the set $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ spans $\mathbb{R}^4$.

Thus, $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ is a basis for $\mathbb{R}^4$.

5. (6 points) Suppose $\{\vec{u}, \vec{v}\}$ is a basis of the eigenspace corresponding to the eigenvalue 0 of a 5×5 matrix A.

(a) Is $\vec{w} = \vec{u} - 2\vec{v}$ an eigenvector of A? If so, find the corresponding eigenvalue. If not, explain why.

$A\bar{w} = A(\bar{u} - 2\bar{v}) = A\bar{u} - 2(A\bar{v}) = \bar{0} - 2(\bar{0})$ since $\bar{u}$ and $\bar{v}$ are eigenvectors corresponding to 0.

So $A\bar{w} = \bar{0} = 0 \cdot \bar{w}$.

Hence $\bar{w}$ is an eigenvector of A and the corresponding eigenvalue is 0.

(b) Find the dimension of Col A.

Dim Nul A = dimension of the eigenspace corresponding to eigenvalue 0

= 2 because $\{\bar{u}, \bar{v}\}$ is a basis of this eigenspace

By the rank theorem,

Dim Nul A + dim Col A = number of columns of A.

$2 + \dim \text{Col } A = 5$

So $\dim \text{Col } A = 5 - 2 = 3$.

6. (8 points) Let $C = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -3 \\ 1 & 2 & 1 \end{bmatrix}$.

(a) Find a basis for Col C.

$C \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. There are pivots in columns one and three.

The pivot columns of C form a basis for Col C.

So a basis for Col $C = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$.

(b) Let $\vec{x} = \begin{bmatrix} -3 \\ -11 \\ 7 \end{bmatrix}$ be a vector in Col C. Find the coordinates of $\vec{x}$ with respect to the basis you found in part (a).

To find the coordinates of $\bar{x}$ with respect to the basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$, we solve the eqn. $c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -11 \\ 7 \end{bmatrix}$.

Augmented matrix is $\begin{bmatrix} 1 & -1 & -3 \\ 2 & -3 & -11 \\ 1 & 1 & 7 \end{bmatrix}$ which reduces to $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$.

Thus $c_1 = 2, c_2 = 5$ are the coordinates of $\bar{x}$ with respect to the basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$.

7. (5 points) Let $B = \begin{bmatrix} -2 & 0 & -9 \\ 0 & 3 & 0 \\ 0 & 5 & 6 \end{bmatrix}$. Find all the eigenvalues of B.

We solve the eqn. $\det(B - \lambda I) = 0$.

$B - \lambda I = \begin{bmatrix} -2-\lambda & 0 & -9 \\ 0 & 3-\lambda & 0 \\ 0 & 5 & 6-\lambda \end{bmatrix}$.

To find $\det(B-\lambda I)$, expand across the first column.

$\det(B-\lambda I) = (-2-\lambda)\det\left(\begin{bmatrix} 3-\lambda & 0 \\ 5 & 6-\lambda \end{bmatrix}\right) - 0 + 0$

$= (-2-\lambda)\left((3-\lambda)(6-\lambda) - 0\right)$

$= (-2-\lambda)(3-\lambda)(6-\lambda)$.

So $\det(B-\lambda I) = 0$ gives $\lambda = -2, 3, 6$.

Thus eigenvalues of B are $-2, 3$ and 6.

8. (5 points) Let $W = \text{Span}\left\{ \underset{\overline{V_1}}{\begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}}, \underset{\overline{V_2}}{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}} \right\}$. Is $\underset{\overline{Z}}{\begin{bmatrix} 2 \\ 6 \\ 4 \\ 0 \end{bmatrix}}$ in $W^\perp$? Explain.

For $\overline{Z}$ to be in $W^\perp$, it must be orthogonal to every vector in a spanning set for W.

$\overline{Z} \cdot \overline{V_1} = 4 + 0 - 4 + 0 = 0$.

$\overline{Z} \cdot \overline{V_2} = -6 + 6 + 0 + 0 = 0$.

Thus, $\overline{Z}$ is in $W^\perp$.