MATH 205A,B LINEAR ALGEBRA - PROF. P. WONG

EXAM II - MARCH 18, 2013

NAME: **Section:**(Circle one) A(1 : 10) B(2 : 40)

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
Total	100	

1. Let

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}.$$

(a)(7 pts) Find the eigenvalues of A.

The eigenvalues of A are the solutions to $\det(A - \lambda I) = 0$. Note that

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1).$$

It follows that the eigenvalues of A are 4 and -1.

(b)(7 pts) For each of the eigenvalue found in (a), determine the corresponding eigenspaces by giving a basis for each such subspace.

For $\lambda = 4, A - 4I = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \sim \begin{bmatrix} -2 & 3 \\ 0 & 0 \end{bmatrix}$. It follows that the corresponding eigenspace is $\left\{ x_2 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} : x_2 \text{ in } \mathbb{R} \right\}$ with $\left\{ \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \right\}$ as a basis.

Similarly for $\lambda = -1$ we have $A + I = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. It follows that the corresponding eigenspace is $\left\{ x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} : x_2 \text{ in } \mathbb{R} \right\}$ with $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ as a basis.

(c)(6 pts) Is A diagonalizable? If so, find an invertible matrix P such that $P^{-1}AP$ is diagonal.

The matrix A is diagonalizble since A has distinct eigenvalues and hence the two vectors $\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are linearly independent eigenvectors. Thus the matrix given by

$$P = \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix}$$

diagonalizes A.

2. Let

$$A = \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 1 & 2 & -1 \\ 2 & 5 & 0 & 0 \end{bmatrix}.$$

(a)(8 pts) Find a basis for the column space Col$A$ of A.

Using row reduction, we have

$$A = \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 1 & 2 & -1 \\ 2 & 5 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & -3 & -6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The first, second, and fourth columns contain pivots so a basis for $\mathrm{Col}A$ **is** $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right\}$**.**

(b)(8 pts) Find a basis for the null space Nul$A$ of A.

From the calculation in (a), the null space of A **is**

$$\mathrm{Nul}A = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Thus a basis for $\mathrm{Nul}A$ **is** $\left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$**.**

(c)(4 pts) What is the rank of A? Justify your answer.

Since A **is a** 3×4 **matrix, the Rank Theorem states that** $\mathrm{rank}A + \dim \mathrm{Nul}A = 4$**. Since** $\dim \mathrm{Nul}A = 1$ **from (b), it follows that** $\mathrm{rank}A = 4 - 1 = 3$**.**

3. (a)(5 pts) Let A be a 4×3 matrix. If $\dim \mathrm{Nul}A = 3$, what is the rank of A? Justify your answer.

The Rank Theorem asserts that $\mathrm{rank}A + \dim \mathrm{Nul}A = n = 3$**. It follows that** $\mathrm{rank}A = 0$**. For instance,** A **could be the zero matrix.**

(b)(5 pts) Suppose B is a 5×5 matrix with eigenvalues $2, -1, 4$ such that the eigenspace corresponding to 2 has dimension 2; the eigenspace corresponding to -1 has dimension 1; and the eigenspace corresponding to 4 has dimension 1. Determine whether B is diagonalizable. Justify your answer.

The matrix B **is NOT diagonalizable because there are four but not five linearly independent eigenvectors.**

(c)(5 pts) Let B be the matrix as in (b). Find $\det(B + I)$, the determinant of the matrix $(B + I)$. Justify your answer.

Since -1 **is an eigenvalue of** B**,** -1 **is a solution to** $\det(B - \lambda I) = 0$ **so** $\det(B + I) = 0$**.**

(d)(5 pts) Let B be the matrix as in (b). What is the dimension of $\mathrm{Col}(B - 2I)$? Justify your answer.

Since 2 **is an eigenvalue of** B**, the null space** $\mathrm{Nul}(B - 2I)$ **is the same as the eigenspace of** B **corresponding to the eigenvalue** $\lambda = 2$**. Thus,** $\dim \mathrm{Nul}(B - 2I) = 2$**. Again, the Rank Theorem asserts that**

$$\dim \mathrm{Col}(B - 2I) + \dim \mathrm{Nul}(B - 2I) = 5.$$

It follows that $\dim \mathrm{Col}(B - 2I) = 5 - 2 = 3$**.**

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by

$$T(x_1, x_2) = (x_1 + x_2, -4x_1 - 3x_2).$$

Consider the basis $\mathcal{B} = \{\begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}\}$.

(a)(8 pts) Find the $\mathcal{B}$-matrix of the transformation T.

Note that $T\left(\begin{bmatrix} 3 \\ -5 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ **and** $T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$**. The coordinate matrix** $P_{\mathcal{B}} = \begin{bmatrix} 3 & -2 \\ -5 & 3 \end{bmatrix}$ **so the inverse is** $P_{\mathcal{B}}^{-1} = \frac{1}{(-1)} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$**. Now,**

$$\left[T\left(\begin{bmatrix} 3 \\ -5 \end{bmatrix}\right)\right]_{\mathcal{B}} = \frac{1}{(-1)} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$\left[T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)\right]_{\mathcal{B}} = \frac{1}{(-1)} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

Therefore the $\mathcal{B}$-matrix of the transformation T is given by $M = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$**.**

(b)(6 pts) Suppose $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find $[T(\vec{x})]_{\mathcal{B}}$, the $\mathcal{B}$-coordinates of $T(\vec{x})$.

First, $[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\vec{x} = \begin{bmatrix} -8 \\ -13 \end{bmatrix}$**. It follows that** $[T(\vec{x})]_{\mathcal{B}} = M[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -8 \\ -13 \end{bmatrix} = \begin{bmatrix} 13 \\ 18 \end{bmatrix}$**.**

[Alternatively, $T(\vec{x}) = \begin{bmatrix} 3 \\ -11 \end{bmatrix}$ **so that** $[T(\vec{x})]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \begin{bmatrix} 3 \\ -11 \end{bmatrix}$**.]**

(c)(6 pts) If $\mathcal{C} = \{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ is another basis, find $[T(\vec{x})]_{\mathcal{C}}$.

Using $\mathcal{C}$, the corresponding coordinate matrix is $P_{\mathcal{C}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **so that its inverse is** $P_{\mathcal{C}}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$**. It follows that**

$$[((\vec{x})]_{\mathcal{C}} = P_{\mathcal{C}}^{-1} \begin{bmatrix} 3 \\ -11 \end{bmatrix} = \begin{bmatrix} 14 \\ -25 \end{bmatrix}.$$

5. (a)(7 pts) Find a basis for the subspace H of $\mathbb{R}^3$ spanned by the vectors $\begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}$.

Note that the first two vectors are scaler multiples of each other whereas the first and the third vectors are linearly independent. In other words, the subspace H can be spanned by the first and the third vectors so a basis for H is $\left\{ \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}$.

(b)(7 pts) Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be the linear transformation given by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(0) \end{bmatrix}$. Find polynomials $\mathbf{p}_1, \mathbf{p}_2$ in $\mathbb{P}_2$ that span the kernel of T.

The kernel $\mathrm{Ker}T$ of T is the set $\{\mathbf{p} : T(\mathbf{p}) = \vec{0}\}$. Since every $\mathbf{p}$ in $\mathbb{P}_2$ is of the form $\mathbf{p}(t) = a + bt + ct^2$, $\mathbf{p}$ lies in $\mathrm{Ker}T$ means that $\mathbf{p}(0) = a$ must be 0. It follows that $\mathrm{Ker}T$ consists of all polynomials from $\mathbb{P}_2$ with zero constant term, or polynomials of the form $bt + ct^2$. Thus, $\mathbf{p}_1 = t, \mathbf{p}_2 = t^2$ span $\mathrm{Ker}T$.

(c)(6 pts) Let λ be an eigenvalue of an $n \times n$ invertible matrix A and $\lambda \neq 0$. Show that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. [Hint: Let $\vec{v}$ be an eigenvector of A corresponding to λ.]

Let $\vec{v}$ be an eigenvector of A corresponding to λ. Since $\lambda \neq 0$, we have $A\vec{v} = \lambda\vec{v}$ or $\dfrac{1}{\lambda}A\vec{v} = \vec{v}$. Now multiply this equality by A^{-1} from the left on both sides, we obtain $\dfrac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$. In other words, the same vector $\vec{v}$ is an eigenvector of A^{-1} corresponding to the eigenvalue $\frac{1}{\lambda}$.