**Math 206**  
**Exam II**  
**3/18/2011**

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**TOTAL**

1. Read the questions carefully.
2. Show all work in the space provided.
3. Clearly indicate your final answer.
4. Be neat.

**GOOD LUCK**
1. Suppose at the point \((x, y, z)\) in \(\mathbb{R}^3\) the temperature \(T\) in degrees F is given by \(T(x, y, z) = xy^2 + z^3\), where \(x\), \(y\) and \(z\) are measured in feet.

1A) Let \(P\) be the point \((2, 1, 3)\). Find the equation of the plane tangent to the level surface of \(T\) at \(P\).

A normal vector for this plane is \(\nabla T \bigg|_{(2,1,3)} = (y^2, 2xy, 3z^2) \bigg|_{(2,1,3)} = (4, 4, 27)\).

So an equation for this plane is 
\[
\overrightarrow{(x-2, y-1, z-3)} \cdot (1, 4, 27) = 0
\]

Or 
\[
(x-2) + 4(y-1) + 27(z-3) = 0
\]

1B) What is the value of \(T\) on this level surface?

\[
T(2,1,3) = 2 \cdot 1^2 + 3^3
\]
\[
= 2 + 27
\]
\[
= 29
\]

1C) At the point \(P\), in what direction is the change in temperature maximized?

in the direction of \(\nabla T \bigg|_{(2,1,3)}\), i.e., in the direction of \((1, 4, 27)\)
2. Let \( T(x, y, z) = x^2 + z^3 \) be the same as in problem (1). Now suppose that an object moves through \( \mathbb{R}^3 \) with position at time \( t \) is given by \( s(t) = (s_1(t), s_2(t), s_3(t)) \), where \( t \) is in minutes, but the component functions \( s_1, s_2 \) and \( s_3 \) are not given explicitly. Suppose at \( t = 2 \) the object is at \( P = (2, 1, 3) \).

2A) Note that the composition \( g = T \circ s \) represents the temperature of the object at time \( t \). Find the formula for \( dg/dt \); express your answer in terms of \( t \) (that is, in terms of \( s_1, s_2 \) and \( s_3 \) and their derivatives).

\[
\begin{align*}
\text{by Jacobians:} & \quad dg/dt = \text{the sole entry of } \left( \frac{dT}{ds} \right)_{t=2} \\
& = \begin{bmatrix}
y^2 & 2xy & 3z^2 \\
\end{bmatrix}
\begin{bmatrix}
ds_1 \\
ds_2 \\
ds_3 \\
\end{bmatrix}
\\
& = y^2 \frac{ds_1}{dt} + 2xy \frac{ds_2}{dt} + 3z^2 \frac{ds_3}{dt} \\
& = (s_2(t))^2 \frac{ds_1}{dt} + 2s_1(t)s_2(t) \frac{ds_2}{dt} + 3s_3(t)^2 \frac{ds_3}{dt}
\end{align*}
\]

2B) Suppose the Jacobian matrix of \( s \) at \( t = 2 \) is \( J_{s}|_{t=2} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 3 & 0 & 6 \end{bmatrix} \); note this is just the velocity vector for the object at \( t = 2 \).

What is the directional derivative of \( T \) at \( P \) in the direction of this velocity vector?

First we need a unit vector in the direction of \( \left[ \begin{array}{c} 2 \\ 3 \\ 6 \end{array} \right] \). Now, \( \left\| \left[ \begin{array}{c} 2 \\ 3 \\ 6 \end{array} \right] \right\| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7 \)

the answer is then \( \nabla T \bigg|_{P} \cdot \frac{1}{7} \left[ \begin{array}{c} 2 \\ 3 \\ 6 \end{array} \right] = \frac{1}{7} \begin{bmatrix} 2+12+162 \end{bmatrix} = \frac{176}{7} \) (\( \approx 25.14 \)°/m)

2C) In degrees per minute, what is the rate of change in temperature experienced by the object at time \( t = 2 \)?

This is \( \left( \frac{dT}{dt} \right) \bigg|_{t=2} \); see 2A's result:

\[
\begin{align*}
& = (s_2(t))^2 \frac{ds_1}{dt} \bigg|_{t=2} + 2s_1(t)s_2(t) \frac{ds_2}{dt} \bigg|_{t=2} + 3s_3(t)^2 \frac{ds_3}{dt} \bigg|_{t=2}
& \text{we're TOLD that}
\end{align*}
\]

at \( t=2 \), the object is at \( P = (2, 1, 3) \), so \( (2, 1, 3) = (s_1(t), s_2(t), s_3(t)) \)

and we get \( \left( \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \right)^2 + 2 \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 3 \end{bmatrix}^2 = 2 + 12 + 162 = 176 \) (°/min)
3. Suppose \( \mathbf{u} = (xy^3, x^2y^2) \) and \( \mathbf{v} = (v_1(r,s,t), v_2(r,s,t)) \). Let \( \mathbf{F} = \mathbf{u} \circ \mathbf{v} \). In terms of \( r, s \) and \( t \), find and simplify as much as possible, \( \partial F_2 / \partial t \).

\[
\frac{\partial F_2}{\partial t} \quad \text{is in the 2nd row, third column of } \quad J F = \begin{array}{c}
J \mathbf{u} \\
J \mathbf{v}
\end{array}
\]

so we need to multiply the 2nd row of \( \text{this} \) by the third column of \( \text{this} \).

\[
\begin{array}{c|c|c}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\hline
2xy^2 & x^2y^2 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\frac{\partial v_1}{\partial r} & \frac{\partial v_1}{\partial s} & \frac{\partial v_1}{\partial t} \\
\hline
\frac{\partial v_2}{\partial r} & \frac{\partial v_2}{\partial s} & \frac{\partial v_2}{\partial t} \\
\end{array}
\]

\[
= 2xy^2 \frac{\partial v_1}{\partial t} + x^2y \frac{\partial v_2}{\partial t}
\]

\[
= 2v_1(r,s,t) (v_2(r,s,t))^2 \frac{\partial v_1}{\partial t} + (v_1(r,s,t))^2 2v_2(r,s,t) \frac{\partial v_2}{\partial t}
\]
4A. Use Maple to find the first, second and third order Taylor polynomials for \( f(x, y) = \frac{32xy}{x + y^2 - 10} \) for the point \( a = (2, 4) \), that is, in powers of \((x - 2)\) and \((y - 4)\). Use of Maple is restricted to finding the required partial derivatives and evaluating them at \((2,4)\). PRINT your Maple work when you are done at the computer.

\[
\begin{array}{c|cccc|cccc}
 f & f_x & f_y & f_{xx} & f_{xy} & f_{yy} & f_{xxy} & f_{xyy} & f_{yyy} \\
32 & 12 & -24 & -3 & -5 & 90 & \frac{9}{8} & \frac{17}{4} & 2 & -102 \\
\end{array}
\]

Assembling all this into the Taylor polynomials yields:

\[
P_1(x,y) = 32 + 12(x-2) - 24(y-4)
\]

\[
P_2(x,y) = P_1(x,y) + \frac{1}{2} \left( -3(x-2)^2 + 2(5)(x-2)(y-4) + 40(y-4)^2 \right)
\]

\[
P_3(x,y) = P_2(x,y) + \frac{1}{3!} \left( \frac{9}{8}(x-2)^3 + 3(\frac{17}{4})(x-2)^2(y-4) + 3(2)(x-2)(y-4)^2 - 102(y-4)^3 \right)
\]

(Note: the Maple file is included as a PDF)

4B. Use your second order Taylor polynomial to approximate \( f(2.1, 4.1) \). Compare the approximation to the actual value by finding the difference. Use five decimal places in your final answers.

\[
P_2(2.1, 4.1) = 32 + 12(0.1) - 24(0.1) + \frac{1}{2} \left( -3(0.1)^2 - 10(0.1)(0.1) + 40(0.1)^4 \right)
\]

\[
= \frac{32 + 1.2 - 2.4 + 0.5(-0.03 - 0.1 + 0.4)}{30.8} = 0.5(0.27)
\]

\[
P_2(2.1, 4.1) = \frac{(32)(2.1)(4.1)}{21 + (4.1)^2 - 10} = 30.92256
\]

\[
\text{difference} = 0.01244
\]
5. Let \( f(x, y) = x + y + \sin(x^2 + y^2) \); then we have the following partial derivatives:

\[
\begin{align*}
  f_x &= 1 + 2 \cos(x^2 + y^2) \cdot x \\
  f_y &= 1 + 2 \cos(x^2 + y^2) \cdot y \\
  f_{xx} &= -4 \sin(x^2 + y^2) \cdot x^2 + 2 \cos(x^2 + y^2) \\
  f_{xy} &= -4 \sin(x^2 + y^2) \cdot xy \\
  f_{yy} &= -4 \sin(x^2 + y^2) \cdot y^2 + 2 \cos(x^2 + y^2)
\end{align*}
\]

5a) A plot of level curves for this function is on the next page. It implies a critical point exists at \((1.47770, 1.47770)\) (to 5 decimal places). Show this point also (very very closely) satisfies the appropriate test to be a critical point. [NOTE WELL: this CP is not necessarily the one you see closest to the origin!]

We must check to see that both \( f_x \) and \( f_y \) are (very close to) 0 at this critical point. Indeed, both \( f_x \) and \( f_y \) equal [by calculator]

\[ -0.0000471 \ldots \text{ which is } \approx 0 \] (given \( 1.47770 \) is right to 5 places)

5b) Use the second derivative test at \((1.47770, 1.47770)\) to see if it’s a local max, min, or saddle point. Show the values of any partial derivatives and other relevant expressions you use; list them all to 5 decimal places.

At this point we find by calculator that

\[ f_{xx} = f_{yy} = 7.54238 \ldots \quad \text{and} \quad f_{xy} = 8.21917 \ldots \quad \text{So,} \]

\[ f_{xx} \text{ is positive and } (f_{xx})(f_{yy}) - (f_{xy})^2 = -10.66666 \ldots \text{ is negative} \]

[i.e., \( |H_1| \) is + and \( |H_2| \) is -] so by the 2nd-derivative test \( f \) has a saddle point at \((1.47770, 1.47770)\).

5c) There seems to be another critical point near \((2.01308, 2.01308)\) (to 5 decimal places). Show this point (very very closely) satisfies the appropriate test to be a critical point.

Now we check to see if \((2.01308, 2.01308)\) "makes" both \( f_x \) and \( f_y \) equal 0; now we find

\[ f_x = f_y = 0.0000009479 \ldots \text{ which is again } \approx 0 \]

5d) Use the second derivative test at \((2.01308, 2.01308)\) to see if it's a local max, min, or saddle point. Show the values of any partial derivatives and other relevant expressions you use; list them all to 5 decimal places.

At this point we find that \( f_{xx} = f_{yy} = -16.1987 \ldots \text{ and } f_{xy} = -15.97020 \ldots \)

So \( f_{xx} \) is negative and \( (f_{xx})(f_{yy}) - (f_{xy})^2 = 15.89666 \ldots \text{ is positive} \)

(i.e., \( |H_1| < 0 \text{ and } |H_2| > 0 \)) so at this critical point we have a local max. (see the figure, two pages from here)
6. Suppose that \( f : \mathbb{R}^3 \to \mathbb{R} \). In each case below, the Hessian matrix given was found at one of the critical points of \( f \). Use the second derivative test, if possible, to decide if the critical point is a local max, min, saddle point.

6A: \[
\begin{bmatrix}
5 & 2 & -2 \\
2 & 2 & 3 \\
-2 & 3 & 20
\end{bmatrix}
\]
\[
|H_1| = 5 \quad |H_2| = 10 - 4 = 6 \quad |H_3| = 43 \text{ (by calculator)}
\]
Sign pattern is \(+ + +\), \(\Rightarrow\) local min

6B: \[
\begin{bmatrix}
-5 & 2 & -2 \\
2 & 2 & 3 \\
-2 & 3 & 20
\end{bmatrix}
\]
\[
|H_1| = -5 \quad |H_2| = -10 - 4 = -14 \quad |H_3| = -267
\]
Sign pattern is \(- - -\), \(\Rightarrow\) saddle point

6C: \[
\begin{bmatrix}
1 & 2 & -2 \\
2 & 4 & 3 \\
-2 & 3 & 20
\end{bmatrix}
\]
\[
|H_1| = 1 \quad |H_2| = 0 \quad \text{... stop! TEST FAILS!}
\]
5. Let \( f(x, y) = x + y + \sin(x^2 + y^2) \); then we have the following partial derivatives:

\[
\begin{align*}
f_x &= 1 + 2 \cos (x^2 + y^2)x \\
f_y &= 1 + 2 \cos (x^2 + y^2)y \\
f_{xx} &= -4 \sin (x^2 + y^2) x^2 + 2 \cos (x^2 + y^2) \\
f_{xy} &= -4 \sin (x^2 + y^2) yx \\
f_{yy} &= -4 \sin (x^2 + y^2) y^2 + 2 \cos (x^2 + y^2)
\end{align*}
\]

In problem 5a, we showed \((1.4777, 1.4777)\) is a critical point; it’s labeled “5a” in the above figure. In problem 5b, we showed that there’s a saddle point at this critical point. The level curves agree with this conclusion since now we know the \(c\)-values the curves represent: the value of \( f \) at this saddle point is about 2.01439, and the level curves imply that in any neighborhood of this point there are points at which \( f \) is bigger than 2.01439 and other points where \( f \) is smaller.

In problem 5c, we showed \((2.01308, 2.01308)\) is also critical point; it’s labeled “5c” in the above figure. In problem 5d, we showed that there’s a local max here. Again, the level curves agree with this. The value of \( f \) at this local max is about 4.9948244.
The following file is for set up and evaluation of Taylor Polynomials

The following line makes sure that if any numeric substitutions have been made for the variables x and y that they are "reset" to variables, so that the partial derivatives can be computed correctly:

\[
x := 'x'; \quad y := 'y';
\]

\[
x := x
\]

\[
y := y
\]

\[
f := \frac{32 (x \cdot y)}{x + y^2 - 10}
\]

Build in the function: we'll always REPEAT the above lines here in case the worksheet is recalculated from THIS point instead of the previous line!

\[
x := 'x'; \quad y := 'y';
\]

\[
f := \frac{32 x y}{x + y^2 - 10}
\]

\[
fx := \frac{\partial}{\partial x} f; \quad fy := \frac{\partial}{\partial y} f
\]

\[
fx := \frac{32 y}{x + y^2 - 10} - \frac{32 x y}{(x + y^2 - 10)^2}
\]

\[
fy := \frac{32 x}{x + y^2 - 10} - \frac{64 y^2}{(x + y^2 - 10)^2}
\]

\[
fx := \frac{\partial}{\partial x} fx; \quad fy := \frac{\partial}{\partial y} fx;
\]

\[
fx := -\frac{64 y}{(x + y^2 - 10)^2} + \frac{64 x y}{(x + y^2 - 10)^3}
\]

\[
fy := -\frac{32 y}{x + y^2 - 10} - \frac{32 x y}{(x + y^2 - 10)^2} + \frac{128 x y^2}{(x + y^2 - 10)^3}
\]

\[
fy := \frac{\partial}{\partial x} fy; \quad fyy := \frac{\partial}{\partial y} fy;
\]

\[
fx := -\frac{192 y}{(x + y^2 - 10)^2} + \frac{256 x y^3}{(x + y^2 - 10)^3}
\]

\[
fyy := -\frac{192 x y}{(x + y^2 - 10)^2} + \frac{256 x^3}{(x + y^2 - 10)^3}
\]

\[
fy := \frac{\partial}{\partial x} fy; \quad fyy := \frac{\partial}{\partial y} fy;
\]

\[
fxx := -\frac{64}{(x + y^2 - 10)^2} + \frac{256 y^2}{(x + y^2 - 10)^3} + \frac{64 x}{(x + y^2 - 10)^3} - \frac{384 x y^2}{(x + y^2 - 10)^4}
\]

\[
fyx := -\frac{192 y}{(x + y^2 - 10)^2} + \frac{384 x y}{(x + y^2 - 10)^3} + \frac{256 y^3}{(x + y^2 - 10)^3} - \frac{768 x y^3}{(x + y^2 - 10)^4}
\]
\[ f_{yyy} := -\frac{192 x}{(x + y^2 - 10)^2} + \frac{1536 x y^2}{(x + y^2 - 10)^3} - \frac{1536 x y^4}{(x + y^2 - 10)^4} \]  

\[ x := 2; y := 4; \]
\[ \text{simplify}(f); \text{simplify}(fx); \text{simplify}(fy); \]
\[ \text{simplify}(fxx); \text{simplify}(fxy); \text{simplify}(fyy); \]
\[ \text{simplify}(fxxx); \text{simplify}(fxy); \text{simplify}(fyy); \]
\[ x := 2 \]
\[ y := 4 \]
\[ 32 \]
\[ 12 \]
\[ -24 \]
\[ -3 \]
\[ -5 \]
\[ 40 \]
\[ \frac{9}{8} \]
\[ \frac{17}{4} \]
\[ 2 \]
\[ -102 \]

\[ \text{evalf}(f); \text{evalf}(fx); \text{evalf}(fy); \]
\[ \text{evalf}(fxx); \text{evalf}(fxy); \text{evalf}(fyy); \]
\[ \text{evalf}(fxxx); \text{evalf}(fxy); \text{evalf}(fyy); \]
\[ 1.125000000 \]
\[ 4.250000000 \]
\[ 2. \]
\[ -102. \]

\[ x := 2; y := 4; f; x := 2.5; y := 4.1; f; \]
\[ x := 2 \]
\[ y := 4 \]
\[ 32 \]
\[ x := 2.5 \]
\[ y := 4.1 \]
\[ 35.23093446 \]