Math 205 Section B
Test 3 (50 points)

Name: Solutions

- Check that you have 6 questions on two pages.
- Show all your work to receive full credit for a problem.

1. (9 points) Let \( A = \begin{bmatrix} -1 & 3 & -2 \\ 0 & 2 & 2 \end{bmatrix} \). Use this matrix to answer the following questions:

(a) Find a basis for \( \text{Col } A \).

\[
A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix}
\]

The first two columns are the pivot columns.

So a basis for \( \text{Col } A = \left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\} \).

(b) \( \text{Col } A \) is a subspace of \( \mathbb{R}^2 \). Is it possible to find a vector in \( \mathbb{R}^2 \) that is not in \( \text{Col } A \)? Explain.

As seen in part (a), \( \dim \text{Col } A = 2 \).

Since \( \text{Col } A \) is a subspace of \( \mathbb{R}^2 \) of dimension 2, and \( \dim \mathbb{R}^2 = 2 \), \( \text{Col } A = \mathbb{R}^2 \).

So it is not possible to find a vector in \( \mathbb{R}^2 \) that is not in \( \text{Col } A \).
2. (7 points) Let \( V \) be a vector space of dimension three. The vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3, \) and \( \vec{v}_4 \) in \( V \) are such that \( \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = V \), and \( \vec{v}_1 + 3\vec{v}_2 - 2\vec{v}_3 + \vec{v}_4 = \vec{0} \). Find a basis for \( V \). Explain how the basis you find satisfies the two conditions in the definition of a basis.

\[
\begin{align*}
\vec{v}_1 &= -3\vec{v}_2 + 2\vec{v}_3 - \vec{v}_4.
\end{align*}
\]

Since \( \vec{v}_1 \) is a linear combination of \( \vec{v}_2, \vec{v}_3, \vec{v}_4 \),
\[
\text{Span} \left\{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \right\} = \text{Span} \left\{ \vec{v}_2, \vec{v}_3, \vec{v}_4 \right\}.
\]

So \( \{\vec{v}_2, \vec{v}_3, \vec{v}_4\} \) is a spanning set for \( V \).

Since \( \dim V = 3 \), any basis of \( V \) contains three vectors.

\( \{\vec{v}_2, \vec{v}_3, \vec{v}_4\} \) is a spanning set with three vectors. So it has to be linearly independent (otherwise we could produce a linearly independent subset that spans \( V \) which would contradict the fact that \( \dim V = 3 \)).

Hence, \( \{\vec{v}_2, \vec{v}_3, \vec{v}_4\} \) is a basis for \( V \). (It is possible to find other bases.)

3. (8 points) A homogeneous system of seven linear equations in eight unknowns has one free variable. Will the system necessarily have a solution for every possible choice of constants on the right sides of the equations? Briefly explain your answer.

Let \( A \) be the coefficient matrix of the homogeneous system.

Then \( A \) is a \( 7 \times 8 \) matrix, with one free variable.

So \( A \) has 7 pivot columns.

This means there is a pivot in each row and so columns of \( A \) span \( \mathbb{R}^7 \). So the system has a solution for every possible choice of constants on the right sides of the equations.

\[\text{OR}\]

\( A \) has one free variable, so \( \dim \text{Nul } A = 1 \).

By the rank theorem, \( 8 = 1 + \text{rank } A \). So \( \text{rank } A = 7 \).

This means \( \dim \text{Col } A = 7 \).

\( \text{Col } A \) is a subspace of \( \mathbb{R}^7 \) of dimension 7.

Hence \( \text{Col } A = \mathbb{R}^7 \).
4. (8 points) Define a linear transformation $T : \mathbb{P}_1 \rightarrow \mathbb{R}$ by $T(p) = p(1)$.

(a) Find a polynomial that spans the kernel of $T$.

Let $\bar{p}(t) = at + bt$. If $\bar{p}$ is in $\text{ker } T$, then $T(\bar{p}) = 0$.

$T(\bar{p}) = \bar{p}(1) = a + b$. So $a + b = 0$, i.e. $a = -b$.

Thus, every polynomial in $\text{ker } T$ is of the form $-b + b t = b (t - 1)$.

So $\text{ker } T = \text{span}\{t - 1\}$.

(b) Is 6 in the range of $T$? Explain.

6 is in the range of $T$, if there is a polynomial $\bar{p}$ such that $T(\bar{p}) = 6$, i.e. $\bar{p}(1) = 6$, i.e. $a + b = 6$.

One possible answer is $a = 1$, $b = 5$. $T(1 + 5t) = 1 + 5 = 6$.

So 6 is in the range of $T$.

5. (8 points) Let $\mathcal{B}$ be the basis of $\mathbb{P}_2$ consisting of the polynomials $1 - t^2$, $t - t^2$, and $2 - 2t + t^2$. Use this basis to answer the following questions:

(a) Let $\bar{p}(t) = 11t - 3t^2$. Find the coordinate vector of $\bar{p}$ relative to $\mathcal{B}$.

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$. $\bar{p} = \begin{bmatrix} 0 \\ 11 \\ -3 \end{bmatrix}$.

Solve the system $\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \\ -3 \end{bmatrix}$.

Solution is $\begin{bmatrix} -16 \\ 27 \\ 8 \end{bmatrix}$.

(b) If $[\bar{q}]_\mathcal{B} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, find the polynomial $\bar{q}$. (Your final answer should be a polynomial and not a column vector.)

$\bar{q} = 1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$\bar{q} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$.

So the polynomial is $-1 + 2t - 2t^2$. 
6. **(10 points)** Let \[ A = \begin{bmatrix} 1 & 5 & -6 & -7 \\ 2 & 4 & 5 & 2 \\ 0 & 0 & -7 & -4 \\ 0 & 0 & 3 & 1 \end{bmatrix} \]. Use this matrix to answer the following questions:

(a) Is \[ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \] an eigenvector of \( A \)? Explain.

\[
A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 5 & -6 & -7 \\ 2 & 4 & 5 & 2 \\ 0 & 0 & -7 & -4 \\ 0 & 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\]

So \[ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \] is an eigenvector of \( A \) corresponding to the eigenvalue 6.

(b) Is 0 an eigenvalue of \( A \)? Explain. If it is an eigenvalue, find a basis and dimension of the corresponding eigenspace.

0 is an eigenvalue of \( A \) if the equation \( A \vec{x} = \vec{0} \) has non-trivial solutions.

\[
\begin{bmatrix} 1 & 5 & -6 & -7 \\ 2 & 4 & 5 & 2 \\ 0 & 0 & -7 & -4 \\ 0 & 0 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

The only solution is the trivial solution.

So 0 is not an eigenvalue of \( A \).

(c) \(-1\) is an eigenvalue of \( A \). The eigenspace corresponding to the eigenvalue \(-1\) is a subspace of \( \mathbb{R}^n \). What is the value of \( n \)? (Do not compute the eigenspace.)

The eigenvectors are in \( \mathbb{R}^4 \).

So \( n = 4 \).