Math 105: Review for Exam II - Solutions

1. **Find dy/dx for each of the following.**

   (a) $\boldsymbol{y = x^2 + 2^x + e^2 + e^{2x} + \ln 2 + \ln(2x) + \arctan 2}$

   $\frac{dy}{dx} = 2x + (\ln 2)2^x + 2e^{2x} + \frac{1}{2x}\cdot 2$ Note that e^2, $\ln 2$, and $\arctan 2$ are constants.

   (b) $\boldsymbol{y = \sqrt{x}\cdot\arctan(5x)}$

   $\frac{dy}{dx} = \frac{1}{2}x^{-1/2}\arctan(5x) + \sqrt{x}\cdot\frac{1}{1+(5x)^2}\cdot 5 = \frac{\arctan(5x)}{2x^{1/2}}\arctan(5x) + \frac{5\sqrt{x}}{1+25x^2}$

   (c) $\boldsymbol{y = \ln(\tan(2^{\cos(x^2)}))}$

   $\frac{dy}{dx} = \frac{1}{\tan(2^{\cos(x^2)})}\cdot\sec^2(2^{\cos(x^2)})\cdot\ln 2(2^{\cos(x^2)})\cdot(-\sin(x^2))\cdot 2x$

   (d) $\boldsymbol{y = \frac{x+e^\pi}{\cos 4 + \sin^5(6x)}}$ Note that e^π and $\cos 4$ are constants.

   $\frac{dy}{dx} = \frac{(1)(\cos 4 + \sin^5(6x)) - (x+e^\pi)(5\sin^4(6x)\cdot\cos(6x)\cdot 6)}{(\cos 4 + \sin^5(6x))^2}$ Recall that $\sin^5(6x) = (\sin(6x))^5$.

2. **Consider the curve defined by $\boldsymbol{x^3 + y^3 = \frac{9}{2}xy}$ (known as the Folium of Descartes).**

   (a) **Find dy/dx.** Use implicit differentiation.

   $$\begin{aligned} 3x^2 + 3y^2\frac{dy}{dx} &= \frac{9}{2}y + \frac{9}{2}x\frac{dy}{dx} \\ 3y^2\frac{dy}{dx} - \frac{9}{2}x\frac{dy}{dx} &= \frac{9}{2}y - 3x^2 \\ \frac{dy}{dx}\left(3y^2 - \frac{9}{2}x\right) &= \frac{9}{2}y - 3x^2 \\ \frac{dy}{dx} &= \frac{\frac{9}{2}y - 3x^2}{3y^2 - \frac{9}{2}x} \end{aligned}$$

   (b) **Verify that the point (1,2) is on the curve above.**

   We must check to see if the values $x = 1$ and $y = 2$ satisfy the equation above.

   $$\begin{aligned} x^3 + y^3 &\overset{?}{=} \frac{9}{2}xy \\ 1^3 + 2^3 &\overset{?}{=} \frac{9}{2}\cdot 1\cdot 2 \\ 9 &\overset{?}{=} 9 \end{aligned}$$

   Thus, the point (1,2) is on the curve.

   (c) **Find the equation of the tangent line at the point (1,2).**

   We want $y = mx + b$.

   $m = \frac{\frac{9}{2}\cdot 2 - 3\cdot 1^2}{3\cdot 2^2 - \frac{9}{2}\cdot 1} = \frac{4}{5}$, so $y = \frac{4}{5}x + b$.

   Now plug in $x = 1$ and $y = 2$ to find b.

   $2 = \frac{4}{5}\cdot 1 + b \Rightarrow \frac{6}{5} = b$

   Therefore, we have $y = \frac{4}{5}x + \frac{6}{5}$.

3. **Evaluate the following limits.**

Throughout this solution, the symbol ★ will stand for whatever notation your instructor prefers for using L'Hopital's Rule on the indeterminate form 0/0; this may be $\overset{\text{“}0/0\text{”}}{=}$ or $\overset{L'H}{=}$ or $\overset{H}{=}$ or $=$ “0/0” or “has the form ‘$\frac{0}{0}$’ and so, by L'Hopital's Rule, is equal to” or something else. The symbol ♡ will serve the same purpose for the indeterminate forms ∞/∞ and $-\infty/\infty$.

(a) $\boldsymbol{\lim\limits_{x\to1} \dfrac{x^3-1}{7-7x}}$ ★ $\lim\limits_{x\to1} \dfrac{3x^2}{-7} = \dfrac{3}{-7} = -\dfrac{3}{7}$

(b) $\boldsymbol{\lim\limits_{x\to0} \dfrac{1-\cos 2x}{3^x}} = \dfrac{0}{1} = 0$ Can't use (and don't need) L'Hopital's Rule!

(c) $\boldsymbol{\lim\limits_{x\to0^+} x^2 \ln x}$

This is of the indeterminate form $0 \cdot (-\infty)$ so we rewrite the function as a fraction in order to use L'Hopital's Rule.

$\lim\limits_{x\to0^+} x^2 \ln x = \lim\limits_{x\to0^+} \dfrac{\ln x}{\frac{1}{x^2}}$ ♡ $\lim\limits_{x\to0^+} \dfrac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim\limits_{x\to0^+} \dfrac{1}{x} \cdot \dfrac{x^3}{-2} = \lim\limits_{x\to0^+} \dfrac{x^2}{-2} = 0$

(d) $\boldsymbol{\lim\limits_{x\to0} \dfrac{1-\cos 4x}{5x^2}}$ ★ $\lim\limits_{x\to0} \dfrac{4\sin 4x}{10x}$ ★ $\lim\limits_{x\to0} \dfrac{16\cos 4x}{10} = \dfrac{16}{10} = \dfrac{8}{5}$

(e) $\boldsymbol{\lim\limits_{x\to\infty} \dfrac{x^2}{2^x}}$ ♡ $\lim\limits_{x\to\infty} \dfrac{2x}{\ln 2 \cdot 2^x}$ ♡ $\lim\limits_{x\to\infty} \dfrac{2}{\ln 2 \cdot \ln 2 \cdot 2^x} = 0$

4. **Suppose that $y = f(t)$ is a solution to the differential equation $y' = \frac{1}{\pi}\arcsin t + y^2$ and that $f\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2}$. Find the equation of the tangent line to f at $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$.**

We want $y = mx + b$.

$$m = \frac{1}{\pi}\arcsin\frac{\sqrt{2}}{2} + \left(\frac{1}{2}\right)^2 = \frac{1}{\pi}\cdot\frac{\pi}{4} + \frac{1}{4} = \frac{1}{2}, \text{ so } y = \frac{1}{2}x + b.$$

Now plug in $x = \frac{\sqrt{2}}{2}$ and $y = \frac{1}{2}$ to find b.

$$\frac{1}{2} = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + b \Rightarrow \frac{1}{2} - \frac{\sqrt{2}}{4} = b$$

Therefore, we have $y = \frac{1}{2}x + \frac{1}{2} - \frac{\sqrt{2}}{4}$.

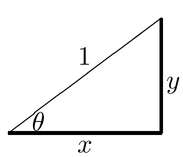
5. **Find the following.**

(a) **an antiderivative of $y = \dfrac{5}{\sqrt{1-9x^2}} + x^3 + \cos(2x) + e^3$**

$$\frac{5\arcsin 3x}{3} + \frac{x^4}{4} + \frac{\sin 2x}{2} + e^3 x + C$$

(b) **$\tan(\arccos x)$ (rewritten as an algebraic expression - no trigonometric functions)**

Let $\theta = \arccos x$. That is, θ is the angle whose cosine is x.



$$x^2 + y^2 = 1^2 \Rightarrow y = \sqrt{1-x^2}$$

$$\tan(\arccos x) = \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} = \frac{\sqrt{1-x^2}}{x}$$

6. **Consider the function $f(x) = x^4e^x$ with domain all real numbers.**

   (a) **Find the x-value(s) of all roots (x-intercepts) of f.**
   The equation $x^4e^x = 0$ means $x^4 = 0$ (that is, $x = 0$) or $e^x = 0$ (no solution), so the only root is at $x = 0$.

   (b) **Find the x- and y-value(s) of all critical points and identify each as a local max, local min, or neither.**

$$f'(x) = 4x^3e^x + x^4e^x$$
$$0 = x^3e^x(4+x)$$
$$\Rightarrow x = 0, -4 \qquad \text{Note that } e^x \text{ is never } 0.$$

	$x < -4$	$-4 < x < 0$	$4 < x$
f'	positive	negative	positive
f	$\nearrow$	$\searrow$	$\nearrow$

y-values: $f(-4) = 256e^{-4} \approx 4.689$, $f(0) = 0$
So, f has a local maximum at $(-4, 256e^{-4})$ and a local minimum at $(0, 0)$.

   (c) **Find the x- and y-value(s) of all global extrema and identify each as a global max or global min.**
   There is a global minimum at $(0, 0)$. There is no global maximum because as $x \to \infty$, $f(x) \to \infty$. Note that as $x \to -\infty$, $f(x) \to 0$. You can verify this by using L'Hopital's Rule on x^4/e^{-x}.

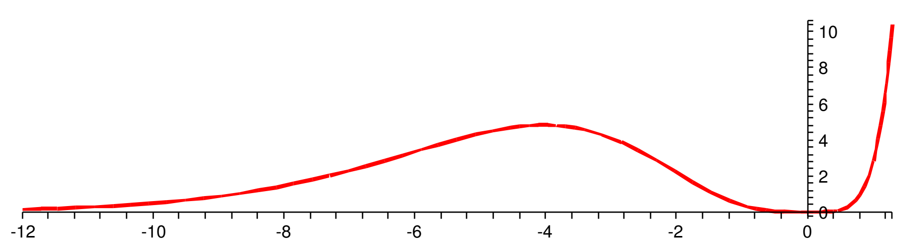
   (d) **Find the x-value(s) of all inflection points.**

$$f''(x) = 12x^2e^x + 4x^3e^x + 4x^3e^x + x^4e^x \quad \text{Use Product Rule on each product in } f'(x) \text{ above.}$$
$$0 = e^x(x^4 + 8x^3 + 12x^2)$$
$$0 = e^x x^2(x^2 + 8x + 12)$$
$$0 = e^x x^2(x+2)(x+6)$$
$$\Rightarrow x = 0, -2, -6$$

	$x < -6$	$-6 < x < -2$	$-2 < x < 0$	$0 < x$
f''	positive	negative	positive	positive
f	concave up	concave down	concave up	concave up

So, the x-values of the inflection points of f are $x = -2$ and $x = -6$ but NOT $x = 0$.

   (e) **Sketch f.**



7. **How would your answers to the previous question change if the domain of f were $[-10, 10]$?**

   There would be a global maximum at $(10, 10^4e^{10})$. (And the graph would be restricted to $-10 \le x \le 10$).

8. **Circle *always*, *sometimes*, or *never* to make each statement below correct.**

   (a) **If $f'(1) = 0$ then f $\boxed{\textit{always}}$/*sometimes*/*never* has a critical point at $x = 1$.**

   A critical point is where f' is 0 or undefined.

   (b) **If $f'(2) = 0$ then f *always*/$\boxed{\textit{sometimes}}$/*never* has a local maximum or local minimum at $x = 2$.**

   f might instead have a terrace point at $x = 2$; we need f' to change sign at $x = 2$ in order to guarantee a local extremum there.

   (c) **If $x = 3$ is a critical point of f, then $f'(3)$ is *always*/$\boxed{\textit{sometimes}}$/*never* 0.**

   It may also be that $f'(3)$ is undefined.

   (d) **If $f''(4) = 0$, then f *always*/$\boxed{\textit{sometimes}}$/*never* has an inflection point at $x = 4$.**

   We need f'' to change sign at $x = 4$ to guarantee an inflection point there.
   For example, if $f(x) = (x-4)^4$ then $f''(4) = 0$ but f has a local minimum rather than an inflection point at $x = 4$.
   However, if $f(x) = (x-4)^3$ then $f''(4) = 0$ and f does have an inflection point at $x = 4$.
   Also see what happens at $x = 0$ in problem 7(d).

   (e) **If f has a global maximum at $x = 5$, then $f'(5)$ is *always*/$\boxed{\textit{sometimes}}$/*never* 0.**

   $f'(5)$ might also be undefined, or $x = 5$ might be an endpoint of the domain.

   (f) **If $f'(6) = 0$ and $f''(6) = -2$, then f $\boxed{\textit{always}}$/*sometimes*/*never* has a local maximum at $x = 6$.**

   If f is concave down with a horizontal slope at $x = 6$, then f must have a local maximum there.

   (g) **If $f'(7) = 0$ and $f''(7) = 0$, then f *always*/$\boxed{\textit{sometimes}}$/*never* has a local extremum at $x = 7$.**

   This means that the second derivative test is inconclusive, so you need to use a different test.
   For example, if $f(x) = (x-7)^4$ then $f'(7) = 0$ and $f''(7) = 0$ and f has a local minimum at $x = 7$.
   However, if $f(x) = (x-7)^3$ then $f'(7) = 0$ and $f''(7) = 0$ and f has an inflection point but not a local extremum at $x = 7$.

9. **The rate of change of a population $P(t)$ of eels is proportional to the size of the population. When the population is 40000, it is growing at a rate of 400 eels per year. At time $t = 0$, the population is 10000.**

   (a) **Write a differential equation whose solution is $P(t)$.**

   Rate of change (P') is (=) proportional to (k) size of population (P) means $P' = kP$.
   What's the value of k? When $P = 40000$, we know $P' = 400$. That is, $400 = k \cdot 40000$, so $k = .01$.
   Thus, we have $P' = .01P$.

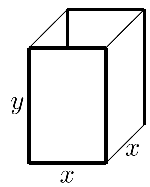
   (b) **Solve your differential equation.**

   The general solution is $P(t) = Ae^{.01t}$.
   What's the value of A? When $t = 0$, we know $P = 10000$. That is, $10000 = Ae^0 = A$, so $A = 10000$.
   Thus, we have $P(t) = 10000e^{.01t}$.

(c) **When will the population reach 60000?**

$$\begin{aligned} 60000 &= 10000e^{.01t} \\ 6 &= e^{.01t} \\ \ln 6 &= \ln e^{.01t} && \text{Take ln of each side.} \\ \ln 6 &= .01t && \text{Recall that } \ln e^z = z. \\ t &= 100 \ln 6 \approx 179.176 \text{ years} \end{aligned}$$

10. **You are designing an 18 $\mathbf{ft^3}$ box that will have a square bottom and no top. The material for the bottom costs 40 cents per square foot and the material for the sides costs 30 cents per square foot. What dimensions give the least total cost? Be sure to show how you know you have found the minimum.**



Goal: minimize cost

Objective function: cost $= C = 40 \cdot x^2 + 30 \cdot 4xy$

We need to get this down to a function of just one variable, so we use the

constraint equation: volume $= 18 = x^2 y$.

Solving for y, we have $y = \dfrac{18}{x^2}$.

Substituting this back into the objective function gives $C = 40 \cdot x^2 + 30 \cdot 4x \cdot \dfrac{18}{x^2} = 40x^2 + \dfrac{2160}{x}$.

Now that we have C as a function of just one variable, we find its minimum.

$$\begin{aligned} C'(x) &= 80x - \frac{2160}{x^2} \\ 0 &= 80x - \frac{2160}{x^2} \\ \frac{2160}{x^2} &= 80x \\ \frac{2160}{80} &= x^3 \\ 3 &= x \end{aligned}$$

Since C' is negative for $0 < x < 3$ and positive for $3 < x$, we know that the minimum occurs at $x = 3$.

And $y = \dfrac{18}{x^2} = \dfrac{18}{3^2} = 2$, so the dimensions are 3 by 3 by 2.

11. **Use the Intermediate Value Theorem to explain why $\boldsymbol{f(x) = x^3 - 4x^2 + 5}$ must have a root somewhere on the interval $\mathbf{[1, 2]}$.**

IVT: If f is continuous on $[a, b]$ and y is a number between $f(a)$ and $f(b)$, then there is a number c between a and b such that $f(c) = y$.

Our function f is continuous on $[1, 2]$. We can compute that $f(1) = 2$ and $f(2) = -3$. Since 0 is a number between 2 and -3, the IVT says there is a number c between 1 and 2 such that $f(c) = 0$; this c is the desired root.

[In plainer English, f is positive at one endpoint and negative at the other. Since f is continuous, the only way its value can go from positive to negative is to go through zero; where f is zero is our root.]