Name: Solutions

Math 206: Winter 2014
Exam 2

Read directions carefully and show all your work. Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. Correct answers without proper justification or those that use unapproved short-cut methods will not receive full credit. If you use a calculator to help find an answer, you must write down enough information on what you have done to make your method understandable.

There are 9 problems and a maximum of 63 points.

Good Luck!

1. (8 points) Determine whether each of the following statements is true or false. BRIEFLY justify your answer.

(a) If $f(x,y)$ is a function such that $f_x(1,-3) < 0$ and $f_y(1,-3) < 0$, then for all unit vectors $\hat{u}$ $f_{\hat{u}}(1,-3) < 0$.

$\boxed{\text{FALSE}}$ for example take $\hat{u} = -\hat{\imath}$

then $f_{\hat{u}}(1,-3) = \nabla f(1,-3) \cdot (-\hat{\imath})$

$= [f_x(1,-3)](-1) = \text{positive}.$

(b) There exists a differentiable function $f(x,y)$ whose tangent plane at $(1,2)$ is $x + 2y = 5$.

$\boxed{\text{FALSE}}$ $x+2y=5$ is a vertical plane

But tangent planes of differentiable functions are never vertical.

2. (3 points) The monthly mortgage payment in dollars, P, for a house is a function of three variables $P = f(A, r, N)$, where A is the amount borrowed in dollars, r is the interest rate, and N is the number of years before the mortgage is paid off. If $\left.\frac{\partial f}{\partial N}\right|_{(100000,7,20)} = \alpha$, interpret α in terms of a mortgage payment. Select all answers that apply. (You do NOT need to show work or justify your answer.)

   (a) We are currently borrowing \$100,000 at 7% interest rate on a 20-year mortgage.

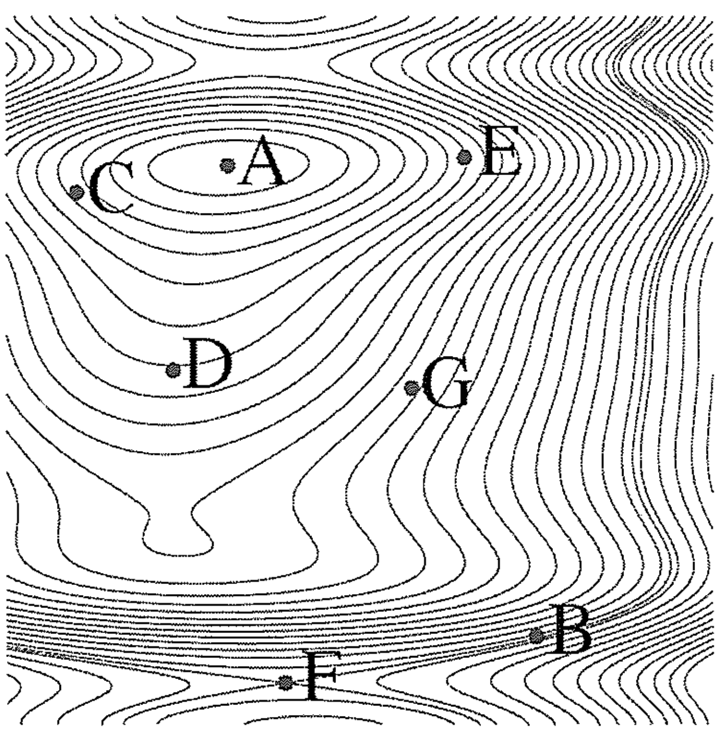
   (b) The monthly payment will go up by approximately \$$\alpha$ for each extra percentage point charged.

   (c) The monthly payment will go up by approximately \$$\alpha$ for each extra dollar we borrow.

   (d) The monthly payment will go up by approximately \$$\alpha$ for each extra year of the mortgage.

3. (8 points) A function $f(x, y)$ of two variables has level curves as shown in the picture. The function values at neighboring level curves differ by 1. Enter A-G in the table below. [No justifications are needed in this problem. Naturally, since there are less points then boxes, some of the points A-G will appear more than once, but each box will only be filled with one letter.]

Enter A-G	is a point, where ...
D	$f_x(x, y) = 0$ and $f_y(x, y) \neq 0$.
E	$f_y(x, y) = 0$ and $f_x(x, y) \neq 0$.
A	$f(x, y)$ has either a max or a min.
F	$f(x, y)$ has a saddle point.
B	the length of the gradient vector of f is largest among all points A-G.
G	$f_{1/\sqrt{2}\hat{\imath}+1/\sqrt{2}\hat{\jmath}}f(x, y) = 0$ and $f_{1/\sqrt{2}\hat{\imath}-1/\sqrt{2}\hat{\jmath}}f(x, y) \neq 0$.
C	$f_{1/\sqrt{2}\hat{\imath}-1/\sqrt{2}\hat{\jmath}}f(x, y) = 0$ and $f_{1/\sqrt{2}\hat{\imath}+1/\sqrt{2}\hat{\jmath}}f(x, y) \neq 0$.
C	the tangent line to the level curve is $x + y = d$ for some constant d.



4. (4 points) f is a function of x and y as follows: $f(x, y)$. Now x and y are themselves functions of r and θ, as follows: $x = g(r, \theta)$ and $y = h(r, \theta)$. Suppose you know that $g(1, \pi/2) = -1$, and $h(1, \pi/2) = 1$. In addition, you are told that

$$\frac{\partial f}{\partial x}(-1,1) = 2, \quad \frac{\partial f}{\partial x}(1,\pi/2) = 3, \quad \frac{\partial f}{\partial y}(-1,1) = 4, \quad \frac{\partial f}{\partial y}(1,\pi/2) = 5$$

$$\frac{\partial x}{\partial r}(1,\pi/2) = 6, \quad \frac{\partial x}{\partial \theta}(1,\pi/2) = 7, \quad \frac{\partial y}{\partial r}(1,\pi/2) = 8, \quad \frac{\partial y}{\partial \theta}(1,\pi/2) = 9$$

Find $\frac{\partial f}{\partial r}(1, \pi/2)$. Show your calculations.

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r}$$

$$\left.\frac{\partial f}{\partial r}\right|_{(1,\pi/2)} = 2 \cdot 6 + 4 \cdot 8 = 12 + 32 = \boxed{44}$$

5. (8 points) Let $f(x, y) = 3xy^2 + 2x^3$.

(a) Use the appropriate partial derivative to find the slope of the cross-section $f(x, 2)$ at the point $(3, 2)$.

$$f_x = 3y^2 + 6x^2$$

$$f_x(3,2) = 3(4) + 6(9) = 12 + 54 = \boxed{66}$$

(b) Use the appropriate partial derivative to determine whether the cross-section $f(x, 2)$ is concave up or down at the point $(3, 2)$.

$$f_{xx} = 12x$$

$f_{xx}(3,2) = 36 > 0 \Rightarrow$ the cross section is concave up

6. (4 points) Let $f(x, y)$ model the time that it takes a rat to complete a maze of length x given that the rat has already run the maze y times. We know $f_y(10, 20) = -5$, $f_x(10, 20) = 1$, and $f(10, 20) = 45$. Use this (and techniques we learned in this class) to estimate $f(11, 18)$.

$$f(x,y) \approx f(10,20) + 1(x-10) - 5(y-20)$$

$$f(11,18) \approx 45 + 1(1) - 5(-2) = \boxed{56}$$

Alternately.

$$df = f_x(10,20)dx + f_y(10,20)dy$$

$$= 1(1) + (-5)(-2) = 11$$

$$\text{so } f(11,18) = 45 + 11 = \boxed{56}$$

7. (4 points) Suppose that as you move away from the point $(2, 0, 1)$, the function $f(x, y, z)$ increases most rapidly in the direction $3\hat{\imath} - \hat{\jmath} + 5\hat{k}$ and the rate of increase of f in this direction is 7. What is $\nabla f(2, 0, 1)$?

$\nabla f(2,0,1)$ in this direction

$$\|3\hat{\imath} - \hat{\jmath} + 5\hat{k}\| = \sqrt{9+1+25} = \sqrt{35}$$

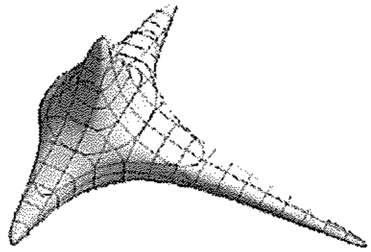
$$\|\nabla f(2,0,1)\| = 7$$

$$\boxed{\nabla f(2,0,1) = \frac{7}{\sqrt{35}}(3\hat{\imath} - \hat{\jmath} + 5\hat{k})}$$

8. (9 points) A seed of "Tribulus terrestris" has the shape

$$x^2 + y^2 + z^2 + x^4y^4 + x^4z^4 + y^4z^4 - 9z = 21.$$

Find the equation of plane tangent to the surface at $(1, 1, 2)$.



Let $t(x,y,z)$
$= x^2 + y^2 + z^2$
$+ x^4y^4 + x^4z^4 + y^4z^4 - 9z$

@ $(1,1,2)$

$t_x = 2x + 4x^3y^4 + 4x^3z^4 \longrightarrow t_x(1,1,2) = 2 + 4 + 4(16) = 70$

$t_y = 2y + 4x^4y^3 + 4y^3z^4 \longrightarrow t_y(1,1,2) = 2 + 4 + 4(16) = 70$

$t_z = 2z + 4x^4z^3 + 4y^4z^3 - 9 \longrightarrow t_z(1,1,2) = 4 + 32 + 32 - 9 = 59$

$$\boxed{70(x-1) + 70(y-1) + 59(z-2) = 0}$$

9. (15 points) Let $f(x, y) = 5 + 3x^2 + 3y^2 + 2y^3 + x^3$.

(a) Find all critical points of f.

(b) Use the second derivative test to classify each critical point you found in (a) as a local maximum, local minimum, or saddle point. If you cannot use the second derivative test to describe the critical points state that and explain why.

(c) Determine whether or not $f(x, y)$ has a global maximum in the xy-plane. Justify your answer.

(a) $f_x = 6x + 3x^2 = 0$ $\quad$ $f_y = 6y + 6y^2 = 0$

$3x(2+x) = 0$ $\quad$ $6y(1+y) = 0$

$x = 0, -2$ $\quad$ $y = 0, -1$

critical points $\quad (0,0), (0,-1), (-2,0), (-2,-1)$

(b)
$$\left.\begin{aligned} f_{xx} &= 6 + 6x \\ f_{xy} &= 0 \\ f_{yy} &= 6 + 12y \end{aligned}\right\} \Rightarrow D = (6+6x)(6+12y)$$

(a,b)	$f_{xx}(a,b)$	D	
$(0,0)$	6	36	local min
$(0,-1)$	6	$6(-6) = -36$	saddle
$(-2,0)$	-6	-36	saddle
$(-2,-1)$	-6	(neg)(neg) $= \boxed{\text{pos}}$	local max

(c) consider $y=0$ cross section: $f(x,0) = 5 + 3x^2 + x^3$

$\lim_{x\to\infty} f(x,0) = \infty$ so $f(x,y)$ gets infinitely large, thus there is no global max.