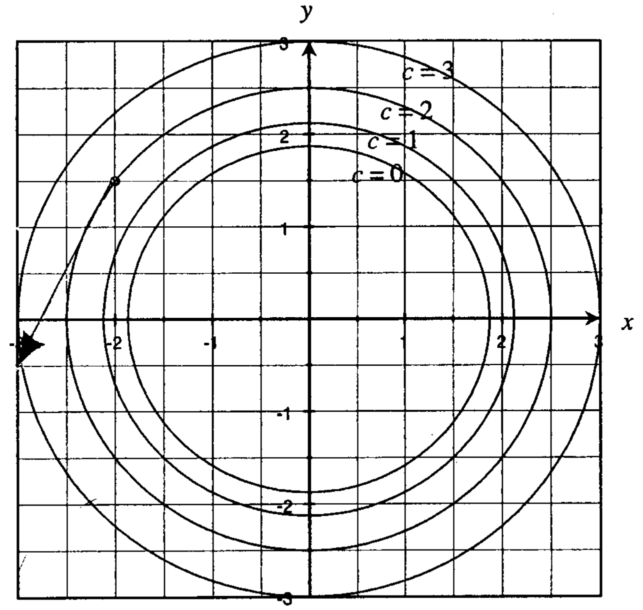
Name: Solutions

Math 206: Winter 2012
Exam 2: March 14

Correct answers accompanied by incorrect or incomplete work will not receive full credit.
Use correct vector notation even in your work. Good Luck!

1. (3 points each) Consider the contour plot for the function $f(x, y)$.



Determine whether the following quantities are *positive, negative,* or *zero.*

(a) $\frac{\partial^2 f}{\partial x^2}(-2, 1.5)$ $\quad \partial f/\partial x(-2,1.5) \approx \frac{-1}{.5} = -2$, $\partial f/\partial x(-1.5,1.5) \approx \frac{-1}{.4} = -\frac{10}{4}$, $\Rightarrow \partial f/\partial x$ gets smaller as x increases

so $\boxed{\text{negative}}$

(b) $D_{\vec{u}} f(-2, 1.5)$ where $\vec{u} = -\hat{\imath} - 2\hat{\jmath}$. $\quad$ a vector in dir'n of $\vec{u}$ is drawn on graph

$\boxed{\text{zero}}$ the vector is roughly tangent, so the slope in that dir'n is close to zero

2. (7 points each) Let $f(x, y) = 7xy^2 - 11x^3y^5$. Compute (in any way you want)

(a) $\frac{\partial^2 f}{\partial x^2}$ $\quad \partial f/\partial x = 7y^2 - 33x^2y^5 \qquad \partial^2 f/\partial x^2 = \boxed{-66xy^5}$

$33 \cdot 5 = 165$

(b) $\frac{\partial^2 f}{\partial y \partial x} = \boxed{14y - 165x^2y^4}$

$||$

(c) $\frac{\partial^2 f}{\partial x \partial y} = \boxed{14y - 165x^2y^4}$

3. (7 points) The temperature in degrees Celsius on the surface of a metal plate is

$$T(x, y) = 20 - 4x^2 - y^2$$

In what direction from the point $(2, -3)$ does the temperature increase most rapidly?

$$\vec{\nabla} T(2,-3) = (-8x, -2y)\Big|_{(2,-3)} = \boxed{(-16, 6)}$$

4. (15 points) $f(x, y) = 2x^3 + 2y^3 - 9xy$ is a surface in $\mathbb{R}^3$.

(a) Calculate $\vec{\nabla} f$. $= (6x^2 - 9y,\ 6y^2 - 9x)$

(b) Write the equation of the line tangent to the level curve $2x^3 + 2y^3 - 9xy = 0$ at the point $(1, 2)$. **Simplify** your answer to the form $y = mx + b$.

$$\vec{\nabla} f(1,2) = (6-18,\ 24-9) = (-12, 15)$$

$$\vec{\nabla} f(1,2) \cdot [(x,y) - (1,2)] = 0$$

$$(-12, 15) \cdot (x-1,\ y-2) = 0$$

$$-12(x-1) + 15(y-2) = 0$$

$$-12x + 12 + 15y - 30 = 0$$

$$15y = 12x + 18$$

$$\boxed{y = \frac{4}{5}x + \frac{6}{5}}$$

5. (15 points) $\vec{f}(t) = \left(1 + t, t^2, \frac{1}{t}\right)$, $1 \le t \le 5$, is the parametric equation for a curve in $\mathbb{R}^3$.

(a) Calculate $\vec{f}'(t)$. $= (1,\ 2t,\ -1/t^2)$

(b) Write the parametric equation of the line tangent to this curve at the point where $t = 1$.

$$\vec{l}(s) = \vec{f}'(1)\, s + \vec{f}(1)$$

$$= (1, 2, -1)s + (2, 1, 1) = \boxed{(s+2,\ 2s+1,\ -s+1) = \vec{l}(s)}$$

6. (20 points) Let $\vec{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\vec{f}(x,y) = \left(\dfrac{2y}{x},\ 3x - xy^2\right)$.

Suppose that $\vec{g} : \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable function such that

$$\vec{g}(1,-1) = \overset{2,3}{(\cancel{3},\cancel{2})} \quad \text{and} \quad D\vec{g}(1,-1) = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}.$$

Let $\vec{h} = \vec{f} \circ \vec{g}$.

(a) Use the chain rule to find $D\vec{h}(1,-1)$.

$$D\vec{h}(1,-1) = D\vec{f}(\vec{g}(1,-1)) \cdot D\vec{g}(1,-1)$$

$$D\vec{f}(\vec{g}(1,-1)) = \begin{bmatrix} -2y/x^2 & 2/x \\ 3-y^2 & -2xy \end{bmatrix}\Bigg|_{(3,2)} = \begin{bmatrix} -3/2 & 1 \\ -6 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} -3/2 & 1 \\ -6 & -12 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 2.5 & 1.5 \\ -54 & 6 \end{bmatrix}}$$

$-3/2 + 8/2 = 5/2$

(b) Use $\vec{h}(1,-1)$ and $D\vec{h}(1,-1)$ to find an approximation for $\vec{h}(0.99,-1.01)$.

$$\vec{h}(.99,-1.01) \approx \vec{h}(1,-1) + D\vec{h}(1,-1)\begin{bmatrix} .99-1 \\ -1.01+1 \end{bmatrix}$$

$$\vec{h}(1,-1) = \vec{f}(2,3) = (3,\ 6-18) = (3,-12)$$

$$\approx \begin{bmatrix} 3 \\ -12 \end{bmatrix} + \begin{bmatrix} 2.5 & 1.5 \\ -54 & 6 \end{bmatrix}\begin{bmatrix} -.01 \\ -.01 \end{bmatrix} = \begin{bmatrix} 3 \\ -12 \end{bmatrix} + \begin{bmatrix} -.04 \\ .48 \end{bmatrix} = \boxed{\begin{bmatrix} 2.96 \\ -11.52 \end{bmatrix}}$$

7. (15 points) Prove that the following function is continuous at $(0,0)$.

$$f(x,y) = \begin{cases} \frac{y^4}{x^4+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

$$0 \le \frac{y^4}{x^4+y^2} = y^2\left(\frac{y^2}{x^4+y^2}\right) \le y^2\left(\frac{x^4+y^2}{x^4+y^2}\right) = y^2$$

$$\lim_{(x,y)\to(0,0)} y^2 = 0$$

and $\displaystyle\lim_{(x,y)\to(0,0)} 0 = 0$

Therefore by the squeeze theorem $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^4}{x^4+y^2} = 0$

hence $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

Thus by the definition of continuous $f(x,y)$ is continuous.

8. (1 point) What is your favorite shape? hyperbolic paraboloid