1. Suppose that V is a vector space and H is a subspace of V and the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_j$ all belong to H. Give the definition of what it means to say that the set $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_j\}$ is a *basis* of H.

2a. Given an $m \times n$ matrix A, the Null Space of A (*ie*, Nul(A)) is defined as the set of solutions of what matrix equation?

2b. In the answer to 2a, is this set is a subspace of $\mathbb{R}^q$ for $q = m$? or $q = n$?

2c. Let $A = \begin{bmatrix} 4 & 7 & 1 & 5 \\ 6 & 10 & 4 & 6 \\ 5 & 8 & 5 & 4 \end{bmatrix}$. Find a basis for the Null Space of A. Show any rref'd matrices involved. Write your final answer in correct basis notation.

3. Suppose that A is a 3×3 matrix. Suppose the following sequence of elementary row operations, labeled Op1, Op2 and Op3, turn A into the identity matrix I_3.

Op1: Rows one and two of A are swapped.

Op2: In the matrix that results from A after Op1 is finished, 5 copies of row one are subtracted from row three.

Op3: In the matrix resulting after Op2 is completed, 3 copies of row two are added to row one.

3a. Let P, M and W be the three elementary matrices which represent Op1, Op2 and Op3, respectively. Explicitly find P, M and W and *label* which is which.

3b. Find A^{-1}, and *explain* how you found it.

3c. Find A. Hint: $(A^{-1})^{-1} = A$. There are several ways available to you for finding this expression.

3d. Find $(A^{-1})^T$.

4a. Let $K = \begin{bmatrix} 6 & 2 & 5 & 2 \\ 3 & 1 & 3 & 3 \\ 9 & 3 & 8 & 5 \end{bmatrix}$; label the column vectors of K as $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, and $\mathbf{k}_4$, respectively. Let $M = \begin{bmatrix} 2 & 5 & 2 & 6 \\ 3 & 3 & 1 & 3 \\ 5 & 8 & 3 & 9 \end{bmatrix}$, so M has the same columns as K, just in a different order. Explain why the column spaces of K and M are the same:

4b. Theorem 6 in section 4.3 tells you that which column vectors of a matrix A form a basis of the Column Space of A?

Answer to 4b:

Now, here are two **facts** that you will find useful:

$$\text{rref}(\ [K|I_3]\ ) = \left[\begin{array}{cccc|ccc} 1 & 1/3 & 0 & -3 & 0 & -8/3 & 1 \\ 0 & 0 & 1 & 4 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right] \qquad \text{rref}([\ M|I_3\ ]) = \left[\begin{array}{cccc|ccc} 1 & 0 & -1/9 & -1/3 & 0 & 8/9 & -1/3 \\ 0 & 1 & 4/9 & 4/3 & 0 & -5/9 & 1/3 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

4c. Use the above **facts** and the answer to (4b) to find a basis for Col(K). Use correct notation for writing a basis.

4d. Now use the **facts** to get a basis for Col(M). (Since the column spaces Col(K) and Col(M) are the same, you now have two different bases for the same subspace).

4d. If you add the column vectors of K you get $\begin{bmatrix} 15 \\ 10 \\ 25 \end{bmatrix}$. This vector is in both column spaces of course. Show how to write it as a LC of the basis vectors you found in (4c). Then show how to write it as a LC of the basis vectors in (4d). Show any rref's you use.

4 BONUS. Through use of super-augmented matrices, the two **facts** give another way to show that Col(K) and Col(M) are the same. Explain what it is.

5. Let H be the subset of all matrices in $M_{2\times 2}$ of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for which $5a + 2c = 0$ and $3a + 4b = 0$.

5A: Give an example of a member of H in which all the entries are different and none of them are 0.

5B: Here's a fact: H is a subspace of $M_{2\times 2}$. Write a correct proof of this fact in the style developed in class. You have three things to show.

6A. Define $T : \mathbb{P}_3 \to \mathbb{P}_2$ by $T(ax^3 + bx^2 + cx + d) = (4a + 5b)x^2 + 6c$. Prove "condition ONE" of the definition of linear transformation holds for T or give a counterexample that shows it does not.

6B: Is T a one-to-one LT? Either prove T is one-to-one, or explain using a counterexample that it is not one-to-one.