MATH106A,B CALCULUS II - PROF. P. WONG

EXAM II - MARCH 13, 2009

NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
Total	100	

1.(10 pts.)(a) Evaluate the indefinite integral

$$\int \frac{x^3}{4+x^2}\ dx.$$

Long division yields

$$\frac{x^3}{4+x^2} = x - \frac{4x}{4+x^2}.$$

Thus,

$$\begin{aligned}\int \frac{x^3}{4+x^2}\ dx &= \int x\ dx - \int \frac{4x}{4+x^2}\ dx \\ &= \frac{x^2}{2} - 2\int \frac{1}{u}\ du \quad \textbf{where } u = 4+x^2 \textbf{ and } du = 2x\ dx \\ &= \frac{x^2}{2} - 2\ln(4+x^2) + C.\end{aligned}$$

Remark: one can do this problem by a simple substitution $u = 4+x^2$ *without using long division.*

(10 pts.)(b) Evaluate the indefinite integral

$$\int \frac{x^2+3x+1}{x(x^2+1)}\ dx.$$

Here, we use the technique of partial fractions. First, write

$$\frac{x^2+3x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

It follows that $A(x^2+1) + x(Bx+C) \equiv x^2+3x+1$**. By equating the coefficients of the like terms, we obtain** $A = 1, B = 0, C = 3$**. Thus,**

$$\begin{aligned}\int \frac{x^2+3x+1}{x(x^2+1)}\ dx &= \int \frac{1}{x} + \frac{3}{x^2+1}\ dx \\ &= \ln|x| + 3\arctan x + C.\end{aligned}$$

2.(10 pts.) Find the **exact** value of the definite integral

$$\int_0^1 \arctan x \; dx.$$

Let $u = \arctan x$ **and** $dv = dx$**. Then** $du = \frac{1}{1+x^2}dx$ **and** $v = x$**. Using IBP, we have**

$$\begin{aligned}\int_0^1 \arctan x \; dx &= x \arctan x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2} \; dx \\ &= x \arctan x\Big|_0^1 - \frac{1}{2}\ln(1+x^2)\Big|_0^1 \quad (\textbf{use substitution } w = 1+x^2) \\ &= (\arctan 1 - 0) - \frac{1}{2}(\ln 2 - \ln 1) = \frac{\pi}{4} - \frac{\ln 2}{2}.\end{aligned}$$

(10 pts.)(b) Find the indefinite integral

$$\int \frac{1}{\sqrt{4x - x^2 + 5}} \; dx.$$

First we rewrite

$$\begin{aligned}4x - x^2 + 5 &= 5 - (x^2 - 4x) = 5 - (x^2 - 4x + 4) + 4 \\ &= 3^2 - (x-2)^2,\end{aligned}$$

using the technique of completing the squares. Then, we use the trigonometric substitution $x - 2 = 3\sin\theta$ **so that** $dx = 3\cos\theta \; d\theta$**. Now, the original integral becomes**

$$\begin{aligned}\int \frac{1}{\sqrt{4x - x^2 + 5}} \; dx &= \int \frac{3\cos\theta \; d\theta}{\sqrt{3^2 - 3^2\sin^2\theta}} \\ &= \int 1 \; d\theta = \theta + C \\ &= \arcsin\left(\frac{x-2}{3}\right).\end{aligned}$$

3. (10 pts.)(a) Let $f(x) = xe^{-x}$. Write down the third-degree Maclaurin polynomial $M_3(x)$ for f.

Note that $f'(x) = e^{-x} + xe^{-x}(-1) = e^{-x}(1-x)$**,** $f''(x) = -e^{-x}(1-x) + e^{-x}(-1) = e^{-x}(x-2)$**, and** $f'''(x) = -e^{-x}(x-2) + e^{-x}(1) = e^{-x}(3-x)$**. Thus,** $f(0) = 0, f'(0) = 1, f''(0) = -2$**, and** $f'''(0) = 3$**. It follows that**

$$M_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!} x^k = x - x^2 + \frac{x^3}{2}.$$

(10 pts.)(b) Let $g(x) = \ln x$. Find the fourth-degree Taylor polynomial $P_4(x)$ for $g(x)$ centered at $x_0 = 1$.

Note that $g'(x) = \frac{1}{x} = x^{-1}$**,** $g''(x) = -x^{-2}$**,** $g'''(x) = 2x^{-3}$**, and** $g^{(4)}(x) = -6x^{-4}$**. Thus,** $g(1) = 0, g'(1) = 1, g''(1) = -1, g'''(1) = 2, g^{(4)}(1) = -6$**. It follows that**

$$P_4(x) = \sum_{k=0}^{4} \frac{g^{(k)}(1)}{k!} x^k = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4.$$

4.(10 pts.)(a) Let $f(x) = \sin(2x)$. What is the maximum possible error, according to Taylor's theorem, committed by using the third-degree Maclaurin polynomial $M_3(x)$ to estimate $f(x)$ for $-\frac{1}{2} \leq x \leq \frac{1}{2}$?

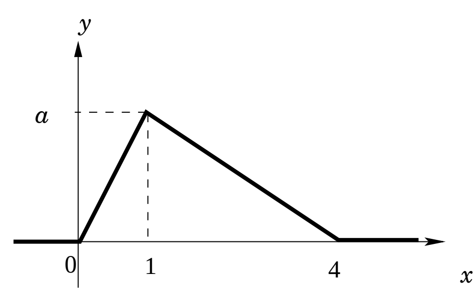
Since $f(x) = \sin(2x)$**, we have** $f'(x) = 2\cos(2x)$**,** $f''(x) = -4\sin(2x)$**,** $f'''(x) = -8\cos(2x)$**, and** $f^{(4)}(x) = 16\sin(2x)$**. It follows that** $|f^{(4)}(x)| \leq 16$ **so we can choose** $K_4 = 16$**. Over the interval** $-\frac{1}{2} \leq x \leq \frac{1}{2}$**,** $|x|^4 \leq \frac{1}{16}$**. Therefore, by Taylor's theorem,**

$$|f(x) - M_3(x)| \leq \frac{K_4|x|^4}{4!} \leq \frac{1}{24}.$$

(10 pts.)(b) Let

$$f(x) = \begin{cases} \text{see graph below,} & \text{if } 0 \leq x \leq 4; \\ 0, & \text{otherwise.} \end{cases}$$

Find a for which $f(x)$ is a probability density function. Justify your answer.



For $f(x)$ **to be a probability density function, (i)** $f(x) \geq 0$ **for all** x **and (ii)** $\int_{-\infty}^{\infty} f(x)\ dx = 1$**. Note that (i) is satisfied. For (ii), it suffices to show that the area under the graph of** f **is equal to** 1 **or the area of the triangle above is** 1**. This means that** $\frac{4a}{2} = 1$ **or** $a = \frac{1}{2}$**.**

5. Determine whether each of the following improper integrals converges or diverges. Justify your answers.

(10 pts.)(a)

$$\int_e^\infty \frac{1}{x(\ln x)^2}\ dx$$

By definition, we have

$$\int_e^\infty \frac{1}{x(\ln x)^2}\ dx = \lim_{b\to\infty}\int_e^b \frac{1}{x(\ln x)^2}\ dx.$$

Using a simple substitution $u = \ln x$, we have

$$\int \frac{1}{x(\ln x)^2}\ dx = \int \frac{1}{u^2}\ du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C.$$

Thus,

$$\int_e^\infty \frac{1}{x(\ln x)^2}\ dx = \lim_{b\to\infty} -\frac{1}{\ln x}\Big|_e^b$$

$$= \lim_{b\to\infty} -\frac{1}{\ln b} + 1 = 0 + 1 = 1.$$

Thus the improper integral converges (to 1).

(10 pts.)(b)

$$\int_2^\infty \frac{x}{\sqrt{x^3-2}}\ dx$$

For $x \geq 2$, $\sqrt{x^3-2} < \sqrt{x^3}$ so we have

$$\int_2^\infty \frac{x}{\sqrt{x^3-2}}\ dx > \int_2^\infty \frac{x}{\sqrt{x^3}}\ dx = \int_2^\infty \frac{1}{x^{1/2}}\ dx.$$

Note that

$$\int_2^\infty \frac{1}{x^{1/2}}\ dx = \int_1^\infty \frac{1}{x^{1/2}}\ dx - \int_1^2 \frac{1}{x^{1/2}}\ dx$$

must diverge using the p-test on the first integral on the right hand side and the fact that the second integral is proper. Thus, by comparison, we conclude that the original improper integral must diverge.