

Exam 2

March 13, 2009

Your Name:

Solutions

There are 6 total problems in this exam. On each problem, you must show all your work, or otherwise thoroughly explain your conclusions. There is always at least one step preceding a final answer. Units may be requested for your final answer; a point deduction will apply if they are omitted.

*On the portion of the exam marked **NO CALCULATOR**, you will be allowed 20 minutes during which your calculator must be closed and put away. If you finish this section early, you may hand in your work early. However, only after you hand in the "no calculators" section will you be permitted to use a calculator.*

You will have 55 minutes to complete this exam.

Question	Point Value	Your Score		
No Calc.	50			
1	40			
2	30			
3	30			
Total	150			

NO CALCULATOR PORTION

Name: Solutions

Math 106-C (Salomone)

Exam 2

Score (50 possible):

Show all your work!

Problem 1-NC. (20 points) Calculate each of the following antiderivatives.

(a) (10 points) $\int \frac{x+3}{x^2-1} dx$ (Partial fractions)

$$\frac{x+3}{x^2-1} = \frac{x+3}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow x+3 = A(x-1) + B(x+1)$$

$$\begin{array}{l} \begin{array}{c} x=1 \\ \swarrow \quad \searrow \\ x+3=2B \quad -1+3=-2A \\ 4=2B \quad -2=-2A \\ 2=B \quad -1=A \end{array} \end{array}$$

$$\int \frac{x+3}{x^2-1} dx = \int \frac{-1}{x+1} + \frac{2}{x-1} dx$$

$$= -\ln|x+1| + 2 \ln|x-1| + C$$

$$= \ln \left| \frac{(x-1)^2}{x+1} \right| + C$$

(b) (10 points) $\int x \arctan \frac{x}{3} dx$ (By parts)

u	v
$\arctan \frac{x}{3}$	x
$\frac{3}{9+x^2}$	$\frac{1}{2}x^2$

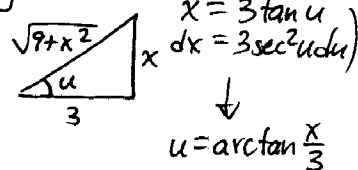
$$= \frac{1}{2}x^2 \arctan \frac{x}{3} - \frac{3}{2} \int \frac{x^2}{9+x^2} dx$$

(Partial fractions:
long division)

$$= \frac{1}{2}x^2 \arctan \frac{x}{3} - \frac{3}{2} \int \left(1 - \frac{9}{9+x^2} \right) dx$$

(Trig subs.: $9+x^2 = 9\sec^2 u$)

$$= \frac{1}{2}x^2 \arctan \frac{x}{3} - \frac{3}{2} \left[x - \int \frac{9}{9\sec^2 u} 3\sec^2 u du \right]$$



$$= \frac{1}{2}x^2 \arctan \frac{x}{3} - \frac{3}{2} \left[x - 3 \int du \right]$$

$$= \frac{1}{2}x^2 \arctan \frac{x}{3} - \frac{3}{2} \left[x - 3u \right] + C$$

$$= \frac{1}{2}x^2 \arctan \frac{x}{3} - \frac{3}{2} \left[x - 3 \arctan \frac{x}{3} \right] + C$$

In problems 2—3, you will find an approximation for the definite integral

$$I = \int_0^1 \frac{\sin x}{x} dx.$$

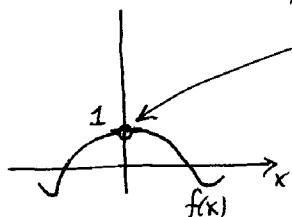
No closed-form antiderivative exists for this function, so the fundamental theorem of calculus cannot be used.

Problem 2-NC. (15 points) Find a 2nd-order Maclaurin polynomial for $f(x) = \frac{\sin x}{x}$, and use it to explain why (and how) f can be considered to be continuous at $x = 0$.

A 'usual suspect' $\rightarrow \sin x \approx x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots$

$$\boxed{\frac{\sin x}{x} \approx 1 - \frac{1}{6}x^2} + \frac{1}{120}x^4 + \dots$$

Since $\frac{\sin x}{x}$ has a continuous Taylor polynomial at $x=0$,



the discontinuity in $f(x)$ is a hole (not a vertical asymptote.)

The Taylor polynomial "fills in" the value $f(0) = \lim_{x \rightarrow 0} f(x) = 1$.

Problem 3-NC. (15 points) Use your answer to #2 to determine (a) a 3rd-order Maclaurin polynomial for an antiderivative $F(x)$ of $f(x)$, and (b) a rational number which approximates I .

Antiderivative of $\frac{\sin x}{x} \approx$ Antiderivative of $1 - \frac{1}{6}x^2 + \dots$

$$\int \frac{\sin x}{x} dx \approx C + x - \frac{1}{18}x^3 + \dots$$

$$\text{So } I = \int_0^1 \frac{\sin x}{x} dx \approx 1 - \frac{1}{18} = \boxed{\frac{17}{18}}$$

(A computer's best approximation of I is 0.946083... compared to $17/18 \approx 0.9\bar{4}$.)

Problem 1. (40 points) This problem concerns the four improper integrals

$$\underbrace{\int_2^{\infty} \frac{1}{\sqrt{x^6 + 6x}} dx}_A$$

$$\underbrace{\int_2^{\infty} \frac{1}{x^3} dx}_B$$

$$\underbrace{\int_0^2 \frac{1}{x^3 - 55x} dx}_C$$

$$\underbrace{\int_2^{\infty} \frac{2 + \sin x}{x^3 + 1} dx}_D$$

(a) (15 points) Only one of these integrals diverges. Which one? Justify your answer.

A: Acts like $\frac{1}{x^3}$ so this converges by the p-test.

B: Same as A.

C: Also acts like $\frac{1}{x^3}$, but here's its VERTICAL asymptote! This DIVERGES by the p-test and the direct comparison

$$\begin{aligned} x^3 - 55x &\leq x^3 \\ \Rightarrow \int_0^2 \frac{1}{x^3 - 55x} &\geq \int_0^2 \frac{1}{x^3} dx \text{ (DIV)} \end{aligned}$$

D: Since $2 + \sin x$ never leaves the interval $[1, 3]$, same as A and B.

(b) (15 points) Rank the three convergent integrals in order from smallest to greatest. Justify your answer.

We know

$$\sqrt{x^6 + 6x} \geq \sqrt{x^6} = x^3$$

$$\boxed{A \leq B \leq D.}$$

And so $\frac{1}{\sqrt{x^6 + 6x}} \leq \frac{1}{x^3}$. Thus $A \leq B$.

Meanwhile, on average $\frac{2 + \sin x}{x^3 + 1} \approx \frac{2}{x^3 + 1}$. And since for all $x > 2$ we have

$$x^3 + 1 \leq 2x^3, \text{ then}$$

$$\frac{2}{x^3 + 1} \geq \frac{2}{2x^3} = \frac{1}{x^3}. \text{ Thus } D \geq B.$$

(c) (10 points) Find an upper bound for the largest integral from part (b). Show your work.

D.

We have

$$x^3 + 1 \geq x^3$$

so $\frac{1}{x^3 + 1} \leq \frac{1}{x^3}$. Also $2 + \sin x \leq 3$. Multiply through to find

$$\frac{2 + \sin x}{x^3 + 1} \leq \frac{3}{x^3}. \text{ Then since } \int_2^{\infty} \frac{3}{x^3} dx = \lim_{t \rightarrow \infty} -\frac{3}{2x^2} \Big|_2^t = \frac{3}{8},$$

we have

$$\int_2^{\infty} \frac{2 + \sin x}{x^3 + 1} dx \leq \frac{3}{8}.$$

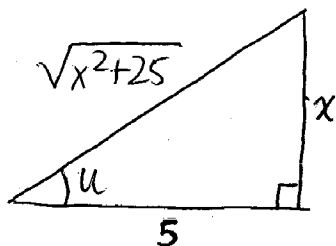
Problem 2. (30 points) Evaluate the antiderivative

$$\int \frac{1}{(x^2 + 25)^{5/2}} dx.$$

Hint: it's hiding something.

Trig substitution!

$$= \int \frac{1}{(\sqrt{x^2 + 25})^5} dx$$



$$\tan u = \frac{x}{5} \rightarrow x = 5 \tan u$$

$$dx = 5 \sec^2 u du$$

$$\cos u = \frac{5}{\sqrt{x^2 + 25}} \rightarrow \sqrt{x^2 + 25} = 5 \sec u$$

$$\sin u = \frac{x}{\sqrt{x^2 + 25}}$$

$$v = \sin u$$

$$dv = \cos u du$$

$$= \int \frac{1}{(5 \sec u)^5} 5 \sec^2 u du$$

$$= \frac{1}{625} \int \frac{1}{\sec^3 u} du$$

$$= \frac{1}{625} \int \cos^3 u du$$

$$= \frac{1}{625} \int (1 - \sin^2 u) \cos u du$$

$$= \frac{1}{625} \int 1 - v^2 dv$$

$$= \frac{1}{625} \left(v - \frac{1}{3} v^3 \right) + C$$

$$= \frac{1}{625} \left(\sin u - \frac{1}{3} \sin^3 u \right) + C$$

$$= \frac{1}{625} \left(\frac{x}{\sqrt{x^2 + 25}} - \frac{1}{3} \frac{x^3}{(x^2 + 25)^{3/2}} \right) + C.$$

Problem 3. (30 points) Let $f(x) = \frac{1}{\sqrt[3]{x+1}} = (x+1)^{-1/3}$

(a) (15 points) Find a 5th-order Maclaurin polynomial for $f(x)$.

	@ $x=0$	Term
$f(x) = (x+1)^{-1/3}$	1	$\frac{1}{0!} x^0 = 1$
$f' = -\frac{1}{3}(x+1)^{-4/3}$	$-1/3$	$\frac{-1/3}{1!} x^1 = -\frac{1}{3}x$
$f'' = \frac{4}{9}(x+1)^{-7/3}$	$4/9$	$\frac{4/9}{2!} x^2 = \frac{2}{9}x^2$
$f''' = \frac{-28}{27}(x+1)^{-10/3}$	$-28/27$	$\frac{-28/27}{3!} x^3 = -\frac{14}{81}x^3$
$f^{IV} = \frac{280}{81}(x+1)^{-13/3}$	$280/81$	$\frac{280/81}{4!} x^4 = \frac{35}{243}x^4$
$f^V = \frac{-3640}{243}(x+1)^{-16/3}$	$-3640/243$	$\frac{-3640/243}{5!} x^5 = -\frac{91}{729}x^5$

$$g_5(x) = 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \frac{35}{243}x^4 - \frac{91}{729}x^5$$

(b) (15 points) Find the exact, best upper bound for the amount of error this polynomial can commit in approximating the true value of $f(3)$.

Hint: don't merely read an approximate K value from a graph. Use the graph to tell you how to find the exact best value for K .

Error is bounded by $\frac{K}{6!} |3-0|^6$ where K bounds $|f^{VI}(x)|$ on $[0,3]$

But $f^{VI}(x) = \frac{58240}{729}(x+1)^{-19/3}$ is a ^{POSITIVE} DECREASING FUNCTION on $(-1, \infty)$ so its maximum value always occurs at left endpoints of subintervals.

Thus $K = f^{VI}(0) = \frac{58240}{729}$ is the best possible bound, and...

... and the error is no more than $\frac{\frac{58240}{729}}{6!} (3)^6 = \frac{728}{9} = \underline{80.8}$.