NAME:

Instruction: Read each question carefully. Explain ALL your work and give reasons to support your answers.

Advice: DON’T spend too much time on a single problem.

<table>
<thead>
<tr>
<th>Problems</th>
<th>Maximum Score</th>
<th>Your Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>22</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
1. Determine whether each of the following improper integrals converges or diverges by comparison. Justify your answers.

(10 pts.)(a) 
\[ \int_1^\infty \frac{1}{(1 + \sqrt{x})^3} \, dx \]

For \( x \geq 1 \), \( (1 + \sqrt{x})^3 > (\sqrt{x})^3 = x^{3/2} \). Thus,
\[
0 < \int_1^\infty \frac{1}{(1 + \sqrt{x})^3} \, dx < \int_1^\infty x^{-3/2} \, dx
\]
\[
= \lim_{b \to \infty} \int_1^b x^{-3/2} \, dx
\]
\[
= \lim_{b \to \infty} -2x^{-1/2} \bigg|_1^b = 2 < \infty.
\]
Hence, the improper integral converges.

(10 pts.)(b) 
\[ \int_2^\infty \frac{\cos^2 x}{x^2} \, dx. \]

Since \( 0 \leq \cos^2 x \leq 1 \), we have \( 0 < \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2} \) when \( x \geq 2 \), and
\[
\int_2^\infty \frac{\cos^2 x}{x^2} \, dx \leq \int_2^\infty \frac{1}{x^2} \, dx < \int_1^\infty \frac{1}{x^2} \, dx,
\]
which converges.

It follows that
\[ \int_2^\infty \frac{\cos^2 x}{x^2} \, dx \]
converges as well.
2. For each of the following improper integrals, evaluate if it exists. Justify your answers.

(10 pts.) (a) \[
\int_{e}^{\infty} \frac{1}{x \ln x} \, dx.
\]

Let \( u = \ln x \) so \( du = \frac{1}{x} \, dx \). When \( x = e \), \( u = 1 \) and when \( x = b \), \( u = \ln b \). Thus,
\[
\int_{e}^{\infty} \frac{1}{x \ln x} \, dx = \lim_{b \to \infty} \left[ \frac{1}{x \ln x} \right]_{e}^{b} = \lim_{b \to \infty} \ln |u| \bigg|_{1}^{\ln b} = \lim_{b \to \infty} \ln |\ln b| - 0 = +\infty.
\]

Hence, the improper integral diverges.

(10 pts.) (b) \[
\int_{-1}^{2} \frac{1}{\sqrt[3]{x}} \, dx
\]

[Hint: this is an improper integral]

The integral is improper at \( x = 0 \). Thus, we write
\[
\int_{-1}^{2} \frac{1}{\sqrt[3]{x}} \, dx = \int_{-1}^{0} x^{-1/3} \, dx + \int_{0}^{2} x^{-1/3} \, dx
\]
\[
= \lim_{b \to 0} \left[ \int_{-1}^{b} x^{-1/3} \, dx + \int_{c}^{2} x^{-1/3} \, dx \right] = \lim_{b \to 0} \left[ \frac{x^{2/3}}{2/3} \right]_{-1}^{b} + \lim_{c \to 0} \left[ \frac{x^{2/3}}{2/3} \right]_{c}^{2} = \lim_{b \to 0} \frac{3}{2} (b^{2/3} - 1) + \lim_{c \to 0} \frac{3}{2} (2^{2/3} - c^{2/3}) = \frac{3}{2} (2^{2/3} - 1).
\]
3. Suppose a function \( f \) satisfies
\[
 f(2) = -1, f'(2) = 0, f''(2) = -2, f'''(2) = 3, f^{(4)}(2) = -4.
\]

(8 pts.) (a) Write down the fourth-degree Taylor polynomial \( P_4(x) \) for \( f \) near \( x = 2 \).

The required polynomial is given by
\[
P_4(x) = -1 + 0 \cdot (x - 2) + \frac{(-2)}{2!} (x - 2)^2 + \frac{3}{3!} (x - 2)^3 + \frac{(-4)}{4!} (x - 2)^4
\]
\[
= -1 - (x - 2)^2 + \frac{1}{2} (x - 2)^3 - \frac{1}{6} (x - 2)^4.
\]

(5 pts.) (b) If \( g(x) = f(x + 2) \), find the third-degree Maclaurin polynomial for \( g(x) \). [Hint: Use part (a).]

Note that \( g^{(n)}(0) = f^{(n)}(2) \). It follows that
\[
M_3(x) = g(0) + g'(0)x + \frac{g''(0)}{2!} x^2 + \frac{g'''(0)}{3!} x^3
\]
\[
= -1 - x^2 + \frac{1}{2} x^3.
\]

(7 pts.) (c) The graph of a function \( h(x) \) is given below. For what value of \( c \) is \( h(x) \) a probability density function?

For the function \( h(x) \) to be a probability density function, we need to check that \( h(x) \geq 0 \) and \( \int_{-\infty}^{\infty} h(x) \, dx = 1 \). The first condition is satisfied. For the second condition, the improper integral gives the area under the graph of \( h(x) \) which is \( (0.5c + c + c) = 2.5c \). It follows that \( 2.5c = 1 \) or \( c = \frac{2}{5} \).
4. (10 pts.) (a) Let \( f(x) = \ln(1 + x) \). Find the third-degree Maclaurin polynomial \( M_3(x) \) for \( f \).

First we need to find \( f^{(n)}(0) \). Note that \( f'(x) = \frac{1}{1+x} = (1 + x)^{-1} \), \( f''(x) = -(1 + x)^{-2} \), and \( f'''(x) = 2(1 + x)^{-3} \). It follows that \( f(0) = \ln 1 = 0 \), \( f'(0) = 1 \), \( f''(0) = -1 \), and \( f'''(0) = 2 \). Thus,

\[
M_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}.
\]

(8 pts.) (b) What is the maximum possible error committed by using \( M_3(x) \) to estimate \( f(x) = \ln(1 + x) \) over the interval \([0, 1]\)?

The maximum error committed is no greater than or equal to \( \frac{K_4}{4!}|x|^4 \). For \( 0 \leq x \leq 1 \), we have \( |x^4| \leq 1 \) and \( K_4 = \max |f^{(4)}(x)| \). From part (a), we have \( f^{(4)}(x) = -6(1+x)^{-4} = \frac{-6}{(1+x)^4} \). Thus, we let \( K_4 = 6 \). In other words, the maximum error committed by using \( M_3(x) \) is no greater than \( \frac{6}{4!} = \frac{1}{6} \).
5. Evaluate each of the following indefinite integral.

(11 pts.) (a) \[ \int \frac{2t}{\sqrt{1 - t^4}} \, dt. \]

Let \( u = t^2 \) so that \( du = 2t \, dt \). Therefore,
\[
\int \frac{2t}{\sqrt{1 - t^4}} \, dt = \int \frac{du}{\sqrt{1 - u^2}} = \arcsin u + C = \arcsin t^2 + C.
\]

(11 pts.) (b) \[ \int \frac{1}{5 + 4x + x^2} \, dx. \]

First, note that \( 5 + 4x + x^2 = 1 + (4 + 4x + x^2) = 1 + (x + 2)^2 \).

Now let \( w = x + 2 \) so that \( dw = dx \) and
\[
\int \frac{1}{5 + 4x + x^2} \, dx = \int \frac{dw}{1 + w^2} = \arctan w + C = \arctan(x + 2) + C.
\]