MATH106A CALCULUS II - PROF. P. WONG

EXAM II - MARCH 9, 2007

NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
Total	100	

1.(10 pts.)(a) Find the exact value of the definite integral

$$\int_1^2 x \ln(x^2)\ dx.$$

Let $u = \ln(x^2)$ **and** $dv = xdx$**. It follows that** $du = \frac{1}{x^2} \cdot 2xdx = \frac{2}{x}dx$ **and** $v = \frac{x^2}{2}$**. Using the techniques of integration by parts, we have**

$$\begin{aligned}\int_1^2 x \ln(x^2)\ dx &= \ln(x^2) \cdot \frac{x^2}{2}\Big|_1^2 - \int_1^2 \frac{x^2}{2} \cdot \frac{2}{x}\ dx \\ &= [2\ln 4 - 0] - \frac{x^2}{2}\Big|_1^2 \\ &= 2\ln 4 - \frac{3}{2}.\end{aligned}$$

(10 pts.)(b) Evaluate the indefinite integral

$$\int \frac{2x}{(x+1)(3x-1)}\ dx.$$

Write $\frac{2x}{(x+1)(3x-1)} = \frac{A}{x+1} + \frac{B}{3x-1}$**. Then,**

$$\begin{aligned}2x &\equiv A(3x-1) + B(x+1) \\ &\equiv (3A+B)x + (-A+B).\end{aligned}$$

So, $2 = 3A + B$ **and** $0 = -A + B$**. It follows that** $A = \frac{1}{2} = B$**.**

Therefore,

$$\begin{aligned}\int \frac{2x}{(x+1)(3x-1)}\ dx &= \frac{1}{2}\int \frac{dx}{x+1} + \int \frac{dx}{3x-1} \\ &= \frac{1}{2}\ln|x+1| + \frac{1}{6}\ln|3x-1| + C.\end{aligned}$$

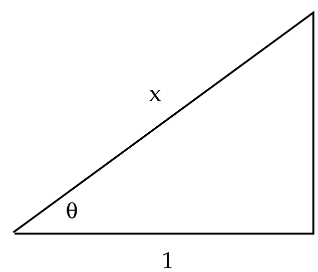
2.(10 pts.) Find the indefinite integral

$$\int \frac{1}{x^2\sqrt{x^2-1}}\ dx.$$

Let $x = \sec\theta$ **so that** $dx = \sec\theta\tan\theta d\theta$**. Now,** $\sqrt{x^2-1} = \tan\theta$ **and** $x^2 = \sec^2\theta$ **so**

$$\int \frac{1}{x^2\sqrt{x^2-1}}\ dx = \int \frac{\sec\theta\tan\theta d\theta}{\sec^2\theta\tan\theta}$$

$$= \int \cos\theta\ d\theta = \sin\theta + C = \frac{\sqrt{x^2-1}}{x} + C.$$



(10 pts.)(b) Find the indefinite integral

$$\int \frac{1}{x^2-2x+5}\ dx.$$

Note that (completing the squares) $x^2-2x+5 = (x^2-2x+1)+4 = (x-1)^2+2^2$**. Now, let** $w = \frac{(x-1)}{2}$ **so** $dw = \frac{dx}{2}$ **or** $dx = 2dw$**. Thus,**

$$\int \frac{1}{x^2-2x+5}\ dx = \int \frac{2dw}{2^2(w^2+1)} = \frac{1}{2}\int \frac{dw}{w^2+1}$$

$$= \frac{1}{2}\arctan w + C = \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + C.$$

3. Suppose a function f satisfies

$$f(1) = 1, f'(1) = -3, f''(1) = 2, f'''(1) = 3.$$

(10 pts.)(a) Write down the third-degree Taylor polynomial $P_3(x)$ for f based at $x_0 = 1$.

The third degree Taylor polynomial is given by

$$\begin{aligned} P_3(x) &= f(1) + f'(1)(x-1) + f''(1)\frac{(x-1)^2}{2!} + f'''(1)\frac{(x-1)^3}{3!} \\ &= 1 - 3(x-1) + (x-1)^2 + \frac{(x-1)^3}{2}. \end{aligned}$$

(5pts.)(b) Suppose it is known that for $0 \le x \le 2$, $|f^{(4)}(x)| \le 0.5$. What is the maximum possible error committed by using $P_3(x)$ to estimate $f(x)$ for $0 \le x \le 2$?

The maximum error is given by $\frac{K_4}{4!}|x-1|^4$ **where** K_4 **is the bound** 0.5**. For** $0 \le x \le 2$**,** $|x-1|^4 \le 1$ **so that the maximum error is** $\frac{0.5}{4!} = \frac{1}{48}$**.**

(5 pts.)(c) Suppose $g(x) = f(x+1)$. Use part (a) to find the third-degree Maclaurin polynomial for g.

Since $g(x) = f(x+1)$**, it follows that** $g^{(k)}(x) = f^{(k)}(x+1)$**. In particular, we have** $g^{(k)}(0) = f^{(k)}(1)$**. Thus,**

$$\begin{aligned} M_3(x) &= g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 \\ &= 1 - 3x + x^2 + \frac{x^3}{2}. \end{aligned}$$

4.(10 pts.)(a) Let $f(x) = \cos(2x)$. Find the fourth-degree Maclaurin polynomial $M_4(x)$ for f.

First, we have $f'(x) = -2\sin(2x)$, $f''(x) = -4\cos(2x)$, $f'''(x) = 8\sin(2x)$ **and** $f^{(4)}(x) = 16\cos(2x)$**. It follows that** $f(0) = 1$, $f'(0) = 0$, $f''(0) = -4$, $f'''(0) = 0$, **and** $f^{(4)}(0) = 16$**. Thus,**

$$M_4(x) = 1 - \frac{4}{2!}x^2 + \frac{16}{4!}x^4 = 1 - 2x^2 + \frac{2}{3}x^4.$$

(10 pts.)(b) Solve the following Initial Value Problem

$$y' = (1 + x^2)e^y, \qquad y(0) = 0.$$

Separating the variables yields

$$\frac{dy}{e^y} = (1 + x^2)dx$$

or

$$\int e^{-y}\ dy = \int (1 + x^2)\ dx.$$

It follows that $-e^{-y} = x + \frac{x^3}{3} + C$**. The initial condition** $y(0) = 0$ **implies that** $C = -1$ **so that** $e^{-y} = 1 - x - \frac{x^3}{3}$**. It follows that**

$$y = -\ln|1 - x - \frac{x^3}{3}|.$$

5. Determine whether each of the following improper integrals converges or diverges. Justify your answers.

(10 pts.)(a)

$$\int_1^\infty \frac{1}{(1+x^3)^2}\, dx$$

[Hint: compare this integral with another improper integral]

For $x \geq 1, (1+x^3) > (x^3)^2 = x^6$**. Thus, for any number** $b \geq 1$**, we have**

$$\int_1^b \frac{1}{(1+x^3)^2}\, dx < \int_1^b \frac{1}{x^6}\, dx = -\frac{1}{5}x^{-5}\Big|_1^b = \frac{1-b^{-5}}{5}.$$

Passing to the limit as $b \to \infty$**, we have**

$$\int_1^\infty \frac{1}{(1+x^3)^2}\, dx = \lim_{b\to\infty}\int_1^b \frac{1}{(1+x^3)^2}\, dx < \lim_{b\to\infty}\frac{1-b^{-5}}{5} = \frac{1}{5}.$$

Thus, $\int_1^\infty \frac{1}{(1+x^3)^2}\, dx$ **converges.**

(10 pts.)(b)

$$\int_0^1 \frac{1}{2x-1}\, dx$$

This integral is improper since the integrand is not defined at $x = 1/2$**. We write**

$$\int_0^1 \frac{1}{2x-1}\, dx = \int_0^{1/2} \frac{1}{2x-1}\, dx + \int_{1/2}^1 \frac{1}{2x-1}\, dx.$$

Now, the improper integral

$$\int_0^{1/2} \frac{1}{2x-1}\, dx = \lim_{b\to 1/2^-} \frac{1}{2x-1}\, dx$$

$$= \lim_{b\to 1/2^-} \frac{1}{2}\ln|2x-1|\Big|_0^b$$

$$= \lim_{b\to 1/2^-} \frac{1}{2}\ln|2b-1| \quad \textbf{which does not exist}.$$

Since one of the two improper integrals does not exist the original improper cannot exist and so it diverges.