NAME: KEY

Math 106B - Exam 2 - March 9, 2007

INSTRUCTIONS: Show all of your work and circle your solutions. Cross out any unnecessary work. Calculators are allowed, but notes and books are not.

Reduction formulas for integrals involving trigonometric expressions are given on the last page of this exam.

1. (10 points) Integrate the following. (**Note**: Your final answer should not contain any function compositions of a trigonometric function with an inverse trigonometric function - for example, $\tan(\arcsin x)$ - as you can simplify these.)

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx.$$

$a^2 - x^2$ case. Let $x = 3\sin t$, so $dx = 3\cos t\,dt$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{(3\sin t)^2}{\sqrt{9-(3\sin t)^2}} \cdot 3\cos t\,dt$$

$$= 3\int \frac{9\sin^2 t\cos t}{\sqrt{9-9\sin^2 t}}\,dt$$

$$= \cancel{3}\int \frac{9\sin^2 t\cos t}{\cancel{3}\cos t}\,dt$$

$$= 9\int \sin^2 t\,dt$$

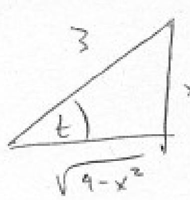
$$= 9\left(\frac{-\sin t\cos t}{2} + \frac{1}{2}\int \sin^0 t\,dt\right) \quad \text{(by formula \#1)}$$

$$= 9\left(\frac{-\sin t\cos t}{2} + \frac{1}{2}\int 1\,dt\right)$$

$$= 9\left(\frac{-\sin t\cos t}{2} + \frac{1}{2}t + c\right)$$

$$= \frac{-9\sin t\cos t}{2} + \frac{9}{2}t + C$$

$\sin t = \frac{x}{3}$



So $\cos t = \frac{\sqrt{9-x^2}}{3}$

$$= \frac{-9\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right)}{2} + \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + C = \boxed{\frac{-x\sqrt{9-x^2}}{2} + \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + C}$$

2. (10 points) Let $f(x) = x \ln x$, and let $P_2(x)$ be the degree-2 Taylor polynomial for $f(x)$ centered at $x_0 = 1$. (**Note**: You do not need to find $P_2(x)$.) According to Taylor's theorem, what is the maximum approximation error committed by $P_2(x)$ on the interval $[0.5, 1.75]$? (**Note**: Explain where your value of K_{n+1} comes from.)

$f(x) = x \ln x$

$f'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1.$

$f''(x) = \frac{1}{x}$

$f'''(x) = -\frac{1}{x^2}.$

For $|f(x) - P_2(x)|$, need K_3.

$|f'''(x)| \leq K_3$ for all x in $[0.5, 1.75]$.

$\left|-\frac{1}{x^2}\right| \leq K_3$, so $\left|\frac{1}{x^2}\right| \leq K_3$. $\left|\frac{1}{x^2}\right|$ is largest when x is smallest.

→ at $x = 0.5$, get

$\frac{1}{(0.5)^2} = 4$. So let $\underline{K_3 = 4}$.

→ $|f(x) - P_2(x)| \leq \frac{K_3}{3!}|x - x_0|^3$

So $|f(x) - P_2(x)| \leq \frac{4}{6}|x-1|^3$. $|x-1|^3$ is largest when $x = 1.75$.

So $|f(x) - P_2(x)| \leq \frac{2}{3}|1.75 - 1|^3 = \frac{2}{3}\left(\frac{3}{4}\right)^3 = \boxed{\frac{9}{32}}.$

3. (10 points) For a function $f(x)$, suppose $P_6(x)$ is the degree-6 Taylor polynomial for $f(x)$ centered at $x_0 = 3$, where

$$P_6(x) = 4 + 2(x-3) + \frac{(x-3)^2}{2} + \frac{2(x-3)^3}{3} + \frac{(x-3)^4}{2} + \frac{2(x-3)^5}{7} + \frac{(x-3)^6}{2}.$$

(a) Determine $f^{(5)}(3)$, or explain why you cannot determine it.

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

So $f^{(5)}(3)$ comes from coefficient of $(x-3)^5$.

From above, that coeff is $\frac{2}{7}$.

So $\frac{2}{7} = \frac{f^{(5)}(3)}{5!}$, so $f^{(5)}(3) = 5!\left(\frac{2}{7}\right) = \boxed{\frac{240}{7}}$

(b) Find the equation of the tangent line to the graph of $y = f(x)$ at $x = 3$.

$f(x_0) = 4$, $f'(x_0) = 2$. $(x_0 = 3)$.

$Pt = (3,4)$, slope $= 2$.

$y - 4 = 2(x-3) \rightarrow \boxed{y = 2x - 2}$

(Also note the Taylor poly of deg 1 is the tan line.

So the tan line is $y = 4 + 2(x-3)$, which is $y = 2x-2$.)

4. (10 points) Use a comparison to determine whether the following improper integral converges or diverges.

$$\int_1^\infty \frac{4}{2\sqrt{x}+5x^3}\,dx.$$

Compare to $\int_1^\infty \frac{4}{5x^3}\,dx$.

① $2\sqrt{x}+5x^3 \geq 5x^3 \geq 0$ for all x in $[1,\infty)$, so

$$\frac{1}{2\sqrt{x}+5x^3} \leq \frac{1}{5x^3}, \text{ so } \frac{4}{2\sqrt{x}+5x^3} \leq \frac{4}{5x^3}.$$

② $\int_1^\infty \frac{4}{5x^3}\,dx = \frac{4}{5}\int_1^\infty \frac{1}{x^3}\,dx$. Integral converges by p-test, $p=3$.

So, since $\int_1^\infty \frac{4}{5x^3}\,dx \geq \int_1^\infty \frac{4}{2\sqrt{x}+5x^3}\,dx \geq 0$, we must have

$\boxed{\int_1^\infty \frac{4}{2\sqrt{x}+5x^3}\,dx \text{ converging, too.}}$

5. (10 points) Evaluate the following integral, or explain why it diverges.

$$\int_2^3 \frac{x}{\sqrt{3-x}}\,dx.$$

Improper at $x=3$. Use limit.

$= \lim_{t\to 3^-} \int_2^t \frac{x}{\sqrt{3-x}}\,dx$. u-sub, $u=3-x$. $(x=3-u)$
$du=-dx$.

$$= \lim_{t\to 3^-}\int_{u=1}^{u=3-t} \frac{3-u}{\sqrt{u}}(-du) = \lim_{t\to 3^-}\int_{u=3-t}^{u=1} 3u^{-1/2}-u^{1/2}\,du = \lim_{t\to 3^-}\left(6u^{1/2}-\frac{2}{3}u^{3/2}\Big]_{u=3-t}^{u=1}\right)$$

$$= \lim_{t\to 3^-}\left(\left(6(1)-\frac{2}{3}(1)\right)-\left(6\sqrt{3-t}-\frac{2}{3}(3-t)^{3/2}\right)\right)$$

$$= \left(6-\frac{2}{3}\right)-(0-0) = \boxed{\frac{16}{3}}$$

6. (10 points) Find all solutions to the differential equation

$$y' = \frac{\sqrt{x}\ln x}{y} + \frac{1}{y}.$$

$$\frac{dy}{dx} = \frac{1}{y}\left(\sqrt{x}\ln x + 1\right).$$

So $\int y\,dy = \int(\sqrt{x}\ln x + 1)\,dx.$ Int by parts.

$$\frac{y^2}{2} = \int \sqrt{x}\ln x\,dx + \int 1\,dx.$$

Aside: $\int \sqrt{x}\ln x\,dx$. $u = \ln x$, $dv = \sqrt{x}\,dx$, $du = \frac{1}{x}dx$, $v = \frac{2}{3}x^{3/2}$

$$= \frac{2}{3}x^{3/2}\cdot\ln x - \int \frac{2}{3}x^{3/2}\cdot\frac{1}{x}\,dx$$

$$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}\,dx$$

$$= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C.$$

So $\frac{y^2}{2} = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + x + C.$

So $y^2 = \frac{4}{3}x^{3/2}\ln x - \frac{8}{9}x^{3/2} + 2x + C.$

$$\boxed{y = \pm\sqrt{\frac{4}{3}x^{3/2}\cdot\ln x - \frac{8}{9}x^{3/2} + 2x + C}}$$

1	2	3	4	5	6	TOTAL (out of 60)

Reduction formulas for integrals containing trigonometric expressions

1. $\displaystyle\int \sin^n(x)\,dx = -\frac{\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$, for $n > 0$.

2. $\displaystyle\int \cos^n(x)\,dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$, for $n > 0$.

3. $\displaystyle\int \tan^n(x)\,dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\,dx$, for $n \neq 1$.

4. $\displaystyle\int \sec^n(x)\,dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$, for $n \neq 1$.

5. $\displaystyle\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$.