MATH 106, Section D

Test # 2

SOLUTIONS

1. (20 points) Find the solution of the IVP. $y' = \left(y^2+2\right)\ln x, \quad y(1) = \sqrt{2}$

$\frac{dy}{dx} = \left(y^2+2\right)\ln x \Longrightarrow \frac{dy}{y^2+2} = \ln x dx \Longrightarrow \frac{1}{2}\int \frac{dy}{\frac{y^2}{2}+1} = \int 1\cdot \ln x dx \Longrightarrow \frac{1}{2}\int \frac{dy}{\left(\frac{y}{\sqrt{2}}\right)^2+1} = \int x'\cdot \ln x dx$

$\boxed{u = \frac{y}{\sqrt{2}} \Longrightarrow du = \frac{1}{\sqrt{2}}dy} \Longrightarrow \frac{1}{\sqrt{2}}\int \frac{du}{u^2+1} = x\ln x - \int x\cdot\frac{1}{x}dx \Longrightarrow \frac{1}{\sqrt{2}}\arctan u = x\ln x - x + C \Longrightarrow$

$\arctan\frac{y}{\sqrt{2}} = x\ln x\sqrt{2} - x\sqrt{2} + C \Longrightarrow y(x) = \sqrt{2}\tan\left(x\ln x\sqrt{2} - x\sqrt{2} + C\right) \Longrightarrow \quad y(1) = \sqrt{2}\tan\left(-\sqrt{2}+C\right) = \sqrt{2}$

$\Longrightarrow \tan\left(-\sqrt{2}+C\right) = 1 \Longrightarrow C = \frac{\pi}{4} + \sqrt{2} \Longrightarrow \boxed{y(x) = \sqrt{2}\tan\left(x\ln x\sqrt{2} - x\sqrt{2} + \frac{\pi}{4} + \sqrt{2}\right)}.$

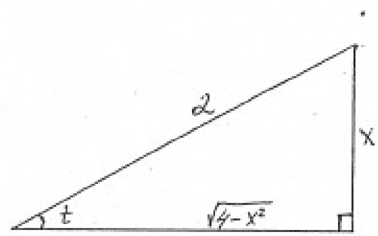
2. (20 points) Evaluate the integral. $\int \frac{dx}{x^4+x^2} = \int \frac{dx}{x^2(x^2+1)}$

$\frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Dx+E}{x^2+1} = \frac{Ax(x^2+1)+B(x^2+1)+(Dx+E)x^2}{x^2(x^2+1)}$

Let $x = 0 \Longrightarrow 1 = B \Longrightarrow \boxed{B=1}$. Coefficients of $x : 0 = A \Longrightarrow \boxed{A=0}$.

Coefficients of $x^3 : 0 = A + D \Longrightarrow \boxed{D=0}$. Coefficients of $x^2 : 0 = B + E \Longrightarrow \boxed{E=-1}$.

Thus $\int \frac{dx}{x^4+x^2} = \int \frac{dx}{x^2} - \int \frac{dx}{x^2+1} = \boxed{-\frac{1}{x} - \arctan x + C}.$



3. (20 points) Evaluate the integral. $\int \frac{x^2}{\sqrt{4-x^2}}dx$

$\boxed{x = 2\sin t, t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \Longrightarrow dx = 2\cos t dt} \Longrightarrow \sqrt{4-x^2} = \sqrt{4-4\sin^2 t} = 2\sqrt{\cos^2 t} = 2\left|\cos t\right| = 2\cos x$

$\int \frac{x^2}{\sqrt{4-x^2}}dx = \int \frac{4\sin^2 t}{2\cos t}2\cos t dt = 4\int \sin^2 t dt = 4\cdot\frac{1}{2}\int(1-\cos 2t)\,dt = 2\int dt - 2\int \cos 2t dt$

$= 2t - 2\cdot\frac{1}{2}\sin 2t + C = 2\arcsin\frac{x}{2} - 2\sin t\cos t + C = 2\arcsin\frac{x}{2} - 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} + C = \boxed{2\arcsin\frac{x}{2} - \frac{x\sqrt{4-x^2}}{2} + C}.$

4. (20 points) Evaluate the integral. $\int x^5 e^{x^2}dx$

$\boxed{u = x^2 \Longrightarrow du = 2xdx} \Longrightarrow \int x^5 e^{x^2}dx = \frac{1}{2}\int \left(x^2\right)^2 e^{x^2}2xdx = \frac{1}{2}\int u^2 e^u du = \frac{1}{2}\int u^2\left(e^u\right)' du$

$= \frac{1}{2}\left[u^2 e^u - \int 2ue^u du\right] = \frac{1}{2}u^2 e^u - \int u\left(e^u\right)' du = \frac{1}{2}u^2 e^u - \left[ue^u - \int e^u du\right] = \frac{1}{2}u^2 e^u - ue^u + \int e^u du$

$= \frac{1}{2}u^2 e^u - ue^u + e^u + C = \boxed{\frac{1}{2}x^4 e^{x^2} - x^2 e^{x^2} + e^{x^2} + C}.$

5. (20 points) Find the fourth-order Taylor polynomial of $f(x) = x^7$ centered at $x_0 = -1$ and determine the upper bound on the approximation error of this polynomial over the interval $[-3, 0]$.

HINT: Suppose that $y = f(x)$ is several times differentiable on the interval I containing x_0. Let K_{n+1} be the minimal positive constant such that $\left|f^{(n+1)}(x)\right| \le K_{n+1}$ on I and $p_n(x)$ be the n-th Taylor polynomial of $f(x)$ centered at x_0, then

$$\left|f(x) - p_n(x)\right| \le \frac{K_{n+1}\left|x - x_0\right|^{n+1}}{(n+1)!}.$$

We know that $p_4(x) = f(-1) + f'(-1)(x+1) + \frac{1}{2}f''(-1)(x+1)^2 + \frac{1}{6}f^{(3)}(-1)(x+1)^3 + \frac{1}{24}f^{(4)}(-1)(x+1)^4$ and the derivatives of $f(x)$ are

$f(x) = x^7$	$f'(x) = 7x^6$	$f''(x) = 42x^5$	$f^{(3)}(x) = 210x^4$	$f^{(4)}(x) = 840x^3$	$f^{(5)}(x) = 2520x^2$
$f(-1) = -1$	$f'(-1) = 7$	$f''(-1) = -42$	$f^{(3)}(-1) = 210$	$f^{(4)}(-1) = -840$	$f^{(5)}(-1) = 2520$

Thus $\boxed{p_4(x) = -1 + 7(x+1) - 21(x+1)^2 + 35(x+1)^3 - 35(x+1)^4}$. Now we need to find critical points of $f^{(5)}(x)$ on $[-3, 0]$: $f^{(6)}(x) = 5040x = 0 \Longrightarrow x = 0$, therefore, we have the following values: $f^{(5)}(0) = 0$, $f^{(5)}(-3) = 22,680 \Longrightarrow K_4 = 22,680 \Longrightarrow \frac{K_4|x-x_0|^5}{5!} = \frac{22,680 \cdot 2^5}{120} = \boxed{6048}$.

6. (20 points) Determine whether the integral is convergent or divergent. If it converges, evaluate it. If it diverges, give reasons.

$$\int_1^9 \frac{dx}{\sqrt[3]{x-9}} = \lim_{t \longrightarrow 9-} \int_1^t \frac{dx}{\sqrt[3]{x-9}} = \lim_{t \longrightarrow 9-} \int_1^t (x-9)^{-1/3}\, dx \qquad \boxed{u = x - 9 \Longrightarrow du = dx \text{ and } [1, t] \longmapsto [-8, t-9]}$$

$$= \lim_{t \longrightarrow 9-} \int_{-8}^{t-9} u^{-1/3} du = \lim_{t \longrightarrow 9-} \frac{1}{2/3} u^{2/3} \Big|_{-8}^{t-9} = \frac{3}{2} \lim_{t \longrightarrow 9-} \left[(t-9)^{2/3} - (-8)^{2/3}\right] = \frac{3}{2} \lim_{t \longrightarrow 9-} \left[(t-9)^{2/3} - 4\right]$$

$= \frac{3}{2} \cdot (-4) = -6$. Thus the original integral converges to $\boxed{-6}$.