1. A Superball is launched straight up into the air from ground level to a height of 80 feet and continues to bounce up and down with each bounce reaching 90% of the height of the previous bounce.

(a) What is the total distance the ball has traveled when it hits the ground for the 10th time?

\[
160 + 160(.9) + 160(.9)^2 + \cdots + 160(.9)^9 = \frac{a(1-r^n)}{1-r} = \frac{160(1-\cdot .9^{10})}{1-.9} \approx 1042.11 \text{ feet}
\]

(b) What is the total distance it will eventually travel?

\[
160 + 160(.9) + 160(.9)^2 + \cdots = \frac{a}{1-r} = \frac{160}{1-.9} = 1600 \text{ feet}
\]

2. Decide if each of the following converges or diverges and justify your answers.

(a) \[\sum_{n=1}^{\infty} \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots\]

\[\lim_{n \to \infty} a_n = \frac{1}{8} \neq 0, \text{ so the } n\text{th term test says this series must } \text{ diverge.}\]

(b) \[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+1)}\]  

Alternating Series Test:

1. Terms alternate in sign
2. \[\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{\ln(n+1)} = 0\]
3. \[\frac{1}{\ln(n+1)} < \frac{1}{\ln(n+1)}\] so \[a_{n+1} < a_n\]

Thus, the series converges.

(c) \[\sum_{n=1}^{\infty} \frac{n^2(3n)!}{28^n(n!)^3}\]  

Ratio Test:

\[\lim_{n \to \infty} \left| \frac{(n+1)^2 \left[3(n+1)\right]!}{28^{n+1} \left[(n+1)!\right]^3} \right| \cdot \frac{28^n}{(3n+3)\cdot \left[3(n+1)\right]!} \cdot \frac{(n)!^3}{28^n \cdot (n!)^3} = \lim_{n \to \infty} \left| \frac{27n^3 + 54n^2 + 33n + 6}{28n^3 + 84n^2 + 84n + 28} \right| = \frac{27}{28} < 1 \Rightarrow \text{ series converges.}\]
\[ d \omega = \frac{1}{x} \, dx. \]

Substitute \( \omega = \ln x \) and \( d \omega = \frac{1}{x} \, dx \). Then,

\[
\sum_{n=2}^{\infty} \frac{1}{n(n \ln n)^5} \text{ Integral Test: } \int_{2}^{\infty} \frac{1}{x(\ln x)^5} \, dx = \lim_{b \to \infty} \left[ \frac{x^{-4}}{-4} \right]_{x=2}^{x=b} = \lim_{b \to \infty} \frac{-1}{4(\ln x)^4} \bigg|_{x=2}^{b} = \lim_{b \to \infty} \left( \frac{1}{4(\ln b)^4} - \frac{1}{4(\ln 2)^4} \right) = \frac{1}{4(\ln 2)^4}. 
\]

Since the integral converges, the series must converge as well (though not to the same value).

Also, the series can be evaluated as

\[
\sum_{n=1}^{\infty} \left( \frac{1}{100 + \frac{1}{n^5}} \right)
\]

\[
\lim_{n \to \infty} a_n = \frac{1}{100} \neq 0, \text{ so by the } n\text{th term test, this series diverges.}
\]

(f) \[
\sum_{n=1}^{\infty} \frac{\sqrt{9n^8 - 1}}{11n^6 + 13n^5}
\]

Comparison:

\[
\frac{\sqrt{9n^8 - 1}}{11n^6 + 13n^5} < \frac{\sqrt{9n^8}}{11n^6} = \frac{3n^4}{11n^6} = \frac{3}{11} \cdot \frac{1}{n^2}.
\]

Thus, our series \( \sum \frac{3}{11} \cdot \frac{1}{n^2} \), which is known to converge (p=2), so we can conclude that our series must converge too.

Therefore, our series must converge too.

(g) \[
\sum_{n=1}^{\infty} \frac{\sqrt{9n^8 + 16n^7}}{11n^6 + 13n^5}
\]

Comparison:

\[
\frac{\sqrt{9n^8 + 16n^7}}{11n^6 + 13n^5} < \frac{\sqrt{9n^8 + 16n^7}}{11n^6} = \frac{5n^4}{11n^6} = \frac{5}{11} \cdot \frac{1}{n^2}.
\]

Thus, our series \( \sum \frac{5}{11} \cdot \frac{1}{n^2} \), which is known to converge (p=2), so we can conclude that our series must converge too.

Therefore, our series must converge too.

3. Find the interval and radius of convergence of \( \sum_{n=1}^{\infty} \frac{2^n(x+3)^n}{5n} \).

\[
\lim_{n \to \infty} \left| \frac{\frac{2^{n+1}(x+3)^{n+1}}{5(n+1)}}{\frac{2^n(x+3)^n}{5n}} \right| = \lim_{n \to \infty} \left| \frac{2^{n+1}}{2^n} \cdot \frac{(x+3)^{n+1}}{(x+3)^n} \cdot \frac{5n}{5(n+1)} \right| = \left| 2 \cdot (x+3) \cdot \frac{1}{5} \right|
\]

So, we get convergence when \( \left| 2(x+3) \right| < 1 \), or \( -1 < 2x+6 < 1 \).

Check endpoints:

\[
\begin{align*}
X = -\frac{7}{2} \text{ gives } & \sum_{n=1}^{\infty} \frac{2^n\left(-\frac{7}{2}+3\right)^n}{5n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{5n} \text{(Alternating Harmonic Series Test)} \quad \text{or} \quad -\frac{7}{2} < X < -\frac{5}{2}; \\
X = -\frac{5}{2} \text{ gives } & \sum_{n=1}^{\infty} \frac{2^n\left(-\frac{5}{2}+3\right)^n}{5n} = \sum_{n=1}^{\infty} \frac{1}{5n} = \frac{1}{5} \sum_{n=1}^{\infty} (\text{div: p}=1) 
\end{align*}
\]

Interval: \(-\frac{7}{2} < X < -\frac{5}{2}\)

Radius: \(\frac{1}{2}\)
4. Find the second degree Taylor polynomial for \( f(x) = \sqrt{x} \) about \( x = 100 \) and use it to estimate \( \sqrt{99} \).

\[
\begin{align*}
\frac{f(100)}{2!} & = \frac{f''(100)}{3!} \cdot (x-100)^2 \\
\frac{f(100)}{20} & = \frac{f''(100)}{2400} \cdot (x-100)^2 \\
\frac{f(100)}{10} & = \frac{f''(100)}{8000} \cdot (x-100)^2 \\
\end{align*}
\]

So, our polynomial is

\[
10 + \frac{1}{20} (x-100) - \frac{1}{8000} (x-100)^2.
\]

To estimate \( \sqrt{99} \), we let \( x = 99 \):

\[
\sqrt{99} \approx \left( 10 + \frac{1}{20} (99-100) - \frac{1}{8000} (99-100)^2 \right) \\
\approx \frac{79599}{8000}.
\]

5. Write out the first three non-zero terms of the Taylor series about \( x = 0 \) for each of the following.

Then write out the complete series for each in summation notation.

(a) \( \cos 3x = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} \ldots \)

Use \( \cos \omega \) series with \( \omega = 3x \).

\[
\cos 3x = \sum_{n=0}^{\infty} \frac{3^{2n} x^{2n} (-1)^n}{2n!}.
\]

(b) \( xe^{-x^2} = x \left( 1 + (-x^2)^2 + \frac{(-x^4)^2}{2!} + \ldots \right) = x - x^3 + \frac{x^5}{2!} \ldots \)

Use \( e^\omega \) series with \( \omega = -x^2 \).

\[
e^{-x^2} = \sum_{n=0}^{\infty} \frac{x^{2n+1} (-1)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n} (-1)^n}{n!}.
\]

(c) \( \frac{1}{1+x^2} = 1 + (-x^2) + (-x^4)^2 + \ldots = 1 - x^2 + x^4 - \ldots \)

Use \( \frac{1}{1-\omega} \) series with \( \omega = -x^2 \).

\[
\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} (-1)^n.
\]

(d) \( \arctan x = \int \frac{dx}{1+x^2} = \int (1 - x^2 + x^4 - \ldots) dx \) [from part (c)]

But \( \arctan 0 = 0 \), so \( C \) must be 0.

\[
\arctan x = \frac{x}{3} + \frac{x^3}{3} + \frac{x^5}{5} - \ldots = \sum_{n=1}^{\infty} \frac{x^{2n-1} (-1)^n}{2n-1}.
\]
6. Use Taylor series to evaluate the following limits.

   \[ \lim_{x \to 0} \frac{1 - 4.5x^2 - \cos 3x}{7x^4} = \lim_{x \to 0} \frac{\frac{-8!}{4!}x^4 + \text{terms with } x^6, x^8}{7x^4} = \frac{-8!}{7 \cdot 24} = -\frac{27}{56} \]

   \[ \lim_{x \to 0} \frac{xe^{-x^2} - x + x^2}{x^5} = \lim_{x \to 0} \left( x - x^3 + \frac{x^5}{2} - \cdots \right) - x + x^3 = \frac{1}{2} \]

7. Use appropriate second-degree Taylor approximations to estimate a solution near \( x = 0 \) to the equation \( e^{5x} - \cos 2x = \sin 4x \).

   \[ e^{5x} \approx 1 + 5x + \frac{25x^2}{2} \]
   \[ \cos(2x) \approx 1 - \frac{(2x)^2}{2!} = 1 - 2x^2 \]
   \[ \sin(4x) \approx 4x \]

   \[ \begin{align*}
   x + 5x + \frac{25x^2}{2} - (1 - 2x^2) &= 4x \\
   x + \frac{29}{2}x^2 &= 0 \\
   x(1 + \frac{29}{2}x) &= 0 \\
   x &= 0, \ -\frac{2}{29} \\
   \end{align*} \]

8. Compute the following derivatives for \( f(x) = \frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots \).

   (a) \( f^{(2004)}(0) \)

   After taking 2004 derivatives and then plugging in \( x = 0 \), the only non-zero term will be the 2004th deriv. of \( x^{2004} \), or \( \frac{2004!}{2004} \).

   (b) \( f^{(2005)}(0) = 0 \)

   After 2005 derivs and plugging in \( x = 0 \), all terms will equal zero.

See old exams and quizzes at http://abacus.bates.edu/~etowne/mathresources.html