NAME: Solutions

Math 205 - Exam 2 - March 8, 2006

Instructions: Show enough work to justify your final answers. When referring to a theorem, give as much information about the theorem as you can.

1. (20 pts.) Consider the matrix $A = \begin{bmatrix} 1 & 2 & -2 & 7 & 6 \\ 2 & 2 & -4 & 10 & 8 \\ 0 & 1 & 0 & 2 & 2 \\ -2 & 2 & 4 & -2 & 6 \end{bmatrix}$, which is row-equivalent to $B = \begin{bmatrix} 1 & 2 & -2 & 7 & 6 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) What is the rank of A?

   (b) What is the dimension of the nullspace of A?

   (c) Find a basis for Col A.

   (d) Find a basis for Row A.

   (e) Find a basis for Nul A.

a) B has 3 pivots, so A has 3 pivots, so $\text{rank}(A) = \boxed{3}$.

b) A has rank 3, and $\dim(\text{Nul}\,A) = n - \text{rank}(A)$
$= 5 - 3 = \boxed{2}$

c) Pivots in B are in columns 1, 2, 5. So basis for Col A is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \\ 2 \\ 6 \end{bmatrix} \right\}.$$

d) Take rows in B that have pivots.

$\rightarrow$ Basis for Row A $= \{(1,2,-2,7,6), (0,1,0,2,2), (0,0,0,0,6)\}$.

e) Solve $A\vec{x} = \vec{0}$. Use augmented matrix

$$[A\ \vec{0}] \sim [B\ \vec{0}] \sim \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$x_1 - 2x_3 + 3x_4 = 0$

$x_2 + 2x_4 = 0$

$x_5 = 0$

x_3, x_4 free.

So $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 - 3x_4 \\ -2x_4 \\ x_3 \\ x_4 \\ 6 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Thus, a basis for Nul A is $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

2. (15 pts.) Consider the matrix $M = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ -1 & -1 & 4 \end{bmatrix}$ and vector $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

(a) Calculate $\det M$.

(b) Is $\mathbf{v}$ in Col M?

(c) Is $\mathbf{v}$ in Nul M?

(d) Is $\mathbf{v}$ an eigenvector of M?

(e) Show that $\lambda = 2$ is an eigenvalue of M and calculate its associated eigenspace.

a) $M \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 4 \\ 0 & -2 & 6 \end{bmatrix}$ (adding row 1 to rows 2 and 3)

$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & 6 \\ 0 & 0 & 4 \end{bmatrix}$. (switching rows 2 and 3).

So $|M| = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 0 & 4 \\ 0 & -2 & 6 \end{vmatrix}$

$= -\begin{vmatrix} 1 & -1 & 2 \\ 0 & -2 & 6 \\ 0 & 0 & 4 \end{vmatrix}$

$= -(1)(-2)(4)$

$\boxed{= 8.}$

b) $M\vec{x} = \vec{v}$ has a sol'n? Aug. matrix $[M\ \vec{v}] = \begin{bmatrix} 1 & -1 & 2 & 3 \\ -1 & 1 & 2 & 1 \\ -1 & -1 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & -2 & 6 & 5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & -2 & 6 & 5 \\ 0 & 0 & 4 & 4 \end{bmatrix}$.

No pivot in last column of $[M\ \vec{v}]$, so there is a solution. (Or note $|M| \neq 0$, so M is invertible, so $\vec{x} = M^{-1}\vec{v}$.) $\boxed{\text{So } \vec{v} \in \text{Col } M.}$

c) $M\vec{v} = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix} \neq \vec{0}$. $\boxed{\text{So } \vec{v} \text{ is not in Nul } M.}$

d) $M\vec{v} = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix} = 2\vec{v}$. $\boxed{\text{So } \vec{v} \text{ is an eigenvector with eigenvalue } \lambda = 2.}$

e) $(M - 2I)\vec{x} = \vec{0}$. Aug matrix is $\begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

So $x_1 = -x_2 + 2x_3$.
x_2, x_3 free.

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

$\boxed{\lambda = 2 \text{ has associated eigenspace } \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}.}$

3. (10 pts.) Let $\mathcal{B} = \{1 - t - t^2, -1 + t - t^2, 2 + 2t + 4t^2\}$ be a set of polynomials in $\mathbb{P}_2$.

(a) Show that $\mathcal{B}$ is a basis for $\mathbb{P}_2$.

(b) For $\mathbf{p}(t) = 3 + t + t^2$, find $[\mathbf{p}]_{\mathcal{B}}$.

(c) Is $\mathcal{B}$ a basis for $\mathbb{P}_3$? Explain.

a) View polys as vectors in $\mathbb{R}^3$. $\rightarrow \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \right\}$.

Independent? $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ -1 & -1 & 4 \end{bmatrix}$ is M from problem 2, so $|M| = 8 \neq 0$,

So ~~since~~ M is invertible, so its columns are independent.

Thus, since $\dim(\mathbb{P}_2) = 3$ and $\mathcal{B}$ is a linearly independent set of 3 polynomials, $\mathcal{B}$ is a basis for $\mathbb{P}_2$.

b) Solve $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ -1 & -1 & 4 \end{bmatrix} \vec{c} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. Use an augmented matrix

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ -1 & 1 & 2 & 1 \\ -1 & -1 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & -2 & 6 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & -2 & 6 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

$$\boxed{\text{So } [\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.}$$

c) $\dim(\mathbb{P}_3) = 4$, so $\mathcal{B}$ would need 4 polynomials in order to be a basis. Sadly, it has only 3. Thus, $\mathcal{B}$ is <u>not</u> a basis for $\mathbb{P}_3$.

4. (10 pts.) Prove the statement: If A is a $m \times n$ matrix, then Nul A is a subspace of $\mathbb{R}^n$.

Nul $A = \{\vec{x} : A\vec{x} = \vec{0}\}$. Certainly Nul A is a <u>subset</u> of $\mathbb{R}^n$ because if A is $m \times n$ and $A\vec{x} = \vec{0}$, then $\vec{x}$ is $n \times 1$. Now we show it's a <u>subspace</u>. Need to verify the 3 properties:

(1) $\vec{0} \in$ Nul A?

$A\vec{0} = \vec{0}$, so $\vec{0}$ is in Nul A.

(2) If $\vec{u}, \vec{v} \in$ Nul A, is $(\vec{u} + \vec{v}) \in$ Nul A?

If $\vec{u}, \vec{v} \in$ Nul A, then $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$. So $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$.

So $\vec{u} + \vec{v} \in$ Nul A.

(3) If $\vec{u} \in$ Nul A, is $c\vec{u} \in$ Nul A for any constant c?

$\vec{u} \in$ Nul A $\Rightarrow A\vec{u} = \vec{0}$. So $A(c\vec{u}) = c(A\vec{u}) = c\vec{0} = \vec{0}$.

So $c\vec{u} \in$ Nul A.

Thus, Nul A is a subspace of $\mathbb{R}^n$!

5. (15 pts.) Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} a & b & c \\ d+2a & e+2b & f+2c \\ g+3a & h+3b & i+3c \end{bmatrix}$, and $C = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{bmatrix}$. Suppose $\det A = 5$.

(a) Find $\det(B)$.

(b) Find $\det(AC)$.

(c) Is AB invertible? Explain.

(d) Are the columns of A linearly independent? Explain.

a) $|B| = |A| = 5$ (adding mults of one row to another leaves determinant unchanged)

b) $|AC| = |A||C| = (5)|C|$. And $|C| = -2|A| = -10$ due to row scaling by 2 and a row flip.

Thus, $|AC| = 5(-10) = -50$.

c) $|AB| = |A||B| = 5 \cdot 5 = 25 \neq 0$, so since the determinant is non-zero, AB is invertible.

d) $|A| \neq 0$, so A is invertible. By Inverse Matrix Theorem, the columns of A are independent.

6. (10 pts.) Suppose A is a $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Prove that each equation $A\mathbf{x} = \mathbf{b}$ has *exactly* one solution.

If $A\vec{x} = \vec{b}$ always has a solution, then by TFAE Thm (in 1.4), A has a pivot in each row. So A has n pivots. So A has a pivot in each column. So $A\vec{x} = \vec{b}$ has no free variables. So $A\vec{x} = \vec{b}$ has at most one solution. And since it always has at least one solution, the conclusion is that $A\vec{x} = \vec{b}$ has exactly one solution.

7. (10 pts.) Let $H = \left\{ \begin{bmatrix} 2a+3b-c+5d \\ 2b+c-6d \\ 2c+d \\ -4c-2d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}$ be a subset of $\mathbb{R}^4$.

(a) Explain why H is a subspace of $\mathbb{R}^4$.

(b) Find a basis for H.

a) $H = \text{Col } A$ for $A = \begin{bmatrix} 2 & 3 & -1 & 5 \\ 0 & 2 & 1 & -6 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -4 & -2 \end{bmatrix}$, and Col A is always a subspace of $\mathbb{R}^m$, which is $\mathbb{R}^4$ in this case.

b) $A \sim \begin{bmatrix} 2 & 3 & -1 & 5 \\ 0 & 2 & 1 & -6 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So A has 3 pivots.

A basis for $H = \text{Col } A$ is then the first 3 columns of A

$\rightarrow \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ -4 \end{bmatrix} \right\}$.

8. (10 pts.) Let A be a 13×15 matrix, and suppose that in trying to solve $A\mathbf{x} = \mathbf{0}$, you find three linearly independent solutions (and there could be more).

(a) Is it possible that $A\mathbf{x} = \mathbf{b}$ always has at least one solution for any vector $\mathbf{b}$ in $\mathbb{R}^{13}$? Why?

(b) Is it possible that $A\mathbf{x} = \mathbf{b}$ ever has exactly one solution? Explain.

3 linearly independent solutions to $A\vec{x} = \vec{0}$ means $\dim(\text{Nul } A) \geq 3$.
So rank $A \leq 12$. (because rank A + $\dim(\text{Nul } A) = n = 15$)

a) rank $A \leq 12$, so A has ≤ 12 pivots, so it doesn't have a pivot in each row, so by TFAE Thm (in 1.4), $\boxed{A\vec{x} = \vec{b} \text{ doesn't always have a solution.}}$

b) There are at least 3 free variables, so if $A\vec{x} = \vec{b}$ has a solution, it must have ∞-ly many. $\boxed{\text{Thus, } A\vec{x} = \vec{b} \text{ cannot have exactly one solution.}}$

1	2	3	4	5	6	7	8	TOTAL