1. Consider the following polynomials in \( \mathbb{P}_2 \):
\[
\begin{align*}
    p_1 &= 10x^2 + 5x + 1, \\
    p_2 &= 9x^2 + 4x, \\
    p_3 &= 11x^2 + x - 7, \\
    p_4 &= x^2 + x + 1.
\end{align*}
\]
Let \( H = \text{Span}\{p_1, p_2, p_3, p_4\} \).

1A. What conditions must \( A, B \) and \( C \) satisfy for \( b = Ax^2 + Bx + C \) to be in \( H \)? Show all matrices used in determining your answer.

This is equivalent to asking what conditions are there on \( A, B, C \) which guarantee a solution \( \alpha, \beta, \gamma \) exists for the equation \( \alpha p_1 + \beta p_2 + \gamma p_3 + \delta p_4 = Ax^2 + Bx + C \).

The equation is also \( \alpha (10x^2 + 5x + 1) + \beta (9x^2 + 4x) + \gamma (11x^2 + x - 7) + \delta (x^2 + x + 1) = Ax^2 + Bx + C \).

Comparing like-powers of \( x \) on both sides gives the system
\[
\begin{align*}
    10\alpha + 9\beta + 11\gamma + \delta &= A, \\
    5\alpha + 4\beta + 1\gamma + \delta &= B, \\
    1\alpha + 0\beta + 1\gamma + \delta &= C.
\end{align*}
\]

Which is represented by
\[
\begin{bmatrix}
    10 & 9 & 11 & 1 & 100 \\
    5 & 4 & 1 & 1 & 100 \\
    1 & 0 & 1 & 1 & 100
\end{bmatrix}
\]

Which has RREF
\[
\begin{bmatrix}
    1 & 0 & -1 & 1 & 100 \\
    0 & 1 & 9 & -1 & 100 \\
    0 & 0 & 0 & 0 & 100
\end{bmatrix}
\]

Which represents a consistent system \( \iff \)
\[
\begin{align*}
    c &= A - \frac{9}{4}B + \frac{7}{4}C \\
    A &= \frac{9}{4}B - \frac{7}{4}C
\end{align*}
\]

1B. Verify that \( d = 24x^2 + 4x - 12 \) satisfies the conditions in 1A.

Does \( 24 = \frac{9}{4} \cdot 4 - \frac{7}{4} \cdot (-12) \) work! Huh?

1C. Find all possible ways to express \( d \) as a linear combination \( \alpha p_1 + \beta p_2 + \gamma p_3 + \delta p_4 \) of \( p_1, p_2, p_3, \) and \( p_4 \).

Using \( A = 24, B = 4, \) and \( C = -12 \), the augmented matrix representing this problem is
\[
\begin{bmatrix}
    10 & 9 & 11 & 1 & 24 \\
    5 & 4 & 1 & 1 & 4 \\
    1 & 0 & -1 & 1 & -12
\end{bmatrix}
\]

Which tells us that
\[
\begin{align*}
    \alpha &= -12 + 7\delta - \frac{5}{\delta}, \\
    \beta &= 16 - 9\delta + \frac{5}{\delta}, \\
    \gamma &= \text{free}, \\
    \delta &= \text{free}
\end{align*}
\]

\[
\begin{bmatrix}
    \alpha \\
    \beta \\
    \gamma \\
    \delta
\end{bmatrix} = \begin{bmatrix}
    -12 \\
    16 \\
    0 \\
    0
\end{bmatrix} + \delta \begin{bmatrix}
    7 \\
    -9 \\
    0 \\
    1
\end{bmatrix}
\]

where \( \gamma \) and \( \delta \) are free.

1D. (Very short yes-or-no answers) Let \( S = \{p_1, p_2, p_3, p_4\} \).

Is \( S \) a linearly independent set? \( \text{NO} \) (then an non-trivial solution for \( \alpha p_1 + \beta p_2 + \gamma p_3 + \delta p_4 = 0 \) exists)

Does \( S \) span (all of) \( \mathbb{P}_3 \)? \( \text{NO} \)

Does \( S \) span (all of) \( H \)? \( \text{Yes} \) of course. see the very start of part 1! (let \( H = \text{span} \{ x^2 + x + 1 \} \))

Is \( S \) a basis for \( H \)? \( \text{NO} \) (not a L.I. set).
2. Let \( M = \begin{bmatrix} 4 & -6 & 5 & 11 & 5 \\ -2 & 3 & -7 & -1 & 2 \\ 2 & -3 & 4 & 4 & 1 \end{bmatrix} \), let \( R \) be the matrix \( \text{RREF}(M) \), let \( b = \begin{bmatrix} 2 \\ 8 \\ -2 \end{bmatrix} \) and \( n = \begin{bmatrix} -15 \\ 8 \\ 9 \\ 3 \\ 6 \end{bmatrix} \).

Fact: the \( \text{RREF} \) of \( [M \mid b] \) is \( \begin{bmatrix} 1 & -3/2 & 0 & 4 & 5/2 & 3 \\ 0 & 0 & 1 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \).

2A. Does \( \text{Col} \ M = \text{Col} \ R \)? Explain. If they are not equal, find a vector in one that’s not in the other.

\( \text{NO} \). Indeed, \( \text{NONE} \) of the column vectors of \( M \) belong to \( \text{Col} \ R \). This is because any linear combination of the column vectors of \( R \) has the form \( \begin{bmatrix} x \\ -4/3 \\ 0 \end{bmatrix} \), and every column vector in \( M \) has a non-zero entry in the bottom position.

2B. Does \( \text{Nul} \ M = \text{Nul} \ R \)? Explain. If they are not equal, find a vector in one that’s not in the other.

\( \text{Yes} \). The solutions of the equations represented by the augmented matrices \( [M \mid d] \) and \( \text{RREF}([M \mid d]) \) are equal.

Let \( d' \) be the vector that \( d \) changes into during the \( \text{RREF} \) process. We are saying the solutions of the equations represented by the augmented matrices \( [M \mid d] \) and \( [R \mid d'] \) are equal. Of course, if \( d \) is the zero vector, then \( d \) and \( d' \) are both \( 0 \); that is, the solutions \( x \) of \( Mx = 0 \) and \( Rx = 0 \) are the same, which means \( \text{Nul} \ M = \text{Nul} \ R \).

2C. Find a basis for \( \text{Col} \ M \).

The pivot columns of \( M \) form a basis of \( \text{Col} \ M \). 
\( \begin{bmatrix} \frac{3}{2} \\ -2 \end{bmatrix}, \begin{bmatrix} \frac{5}{2} \\ 8 \end{bmatrix} \) is a basis of \( \text{Col} \ M \).

2D. Find a basis for \( \text{Col} \ R \).

Its pivot columns are a basis; so \( \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) is a basis.

2E. Show how to express \( b \) as a linear combination of the basis vectors in 2C.

\( \text{We find} \text{RREF} \left( \begin{bmatrix} -2 & 3/2 & 2 \\ 2 & -4 & 2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -2 \end{bmatrix} \) (from the Fact above). Which tells us that \( \begin{bmatrix} 2 \\ 8 \end{bmatrix} = 3 \begin{bmatrix} -2 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 5/2 \\ -4 \end{bmatrix} \).

2F. Find a basis for \( \text{Nul} \ M \).

The Fact tells us that \( Mx = 0 \) for \( x = \begin{bmatrix} 3/2x_2 - 4x_4 - 5/2x_5 \\ x_2 \\ x_4 + x_5 \\ x_4 - x_5 \end{bmatrix} \). 
\( x_2, x_4, x_5 \) are free. 
So the set \( \begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -5/2 \\ 0 \\ 1 \end{bmatrix} \) spans \( \text{Nul} \ M \).

2G. Find a basis for \( \text{Nul} \ R \).

Since \( \text{Nul} \ M = \text{Nul} \ R \), any basis of \( \text{Nul} \ M \) is a basis of \( \text{Nul} \ R \). In particular the basis in \( 2F \) is a basis of \( \text{Nul} \ R \).

2H. It's a fact that \( n \in \text{Nul} M \). Express \( n \) as a linear combination of your basis vectors in \( 2F \).

It's easy to see that \( \begin{bmatrix} -15 \\ 8 \\ 9 \\ 3 \\ 6 \end{bmatrix} = 8 \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} -5/2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \) (4 of the 6 column 1's in rows 2, 4 and 5 of the basis vectors).
3. Consider the vector space \( \mathbb{F} \) of all continuous functions \( f : \mathbb{R} \rightarrow \mathbb{R} \). Let \( H \) be the subset of functions in \( \mathbb{F} \) that have horizontal tangents when they intersect the vertical line \( x = 1 \). Three typical members of \( H \) are drawn in the figure to the right.

3A. For any member of \( f \) in \( H \), what is \( f'(1) \)? \( f'(1) = 0 \).

3B. Give a well-written proof that \( H \) is a subspace of \( \mathbb{F} \).

1. Is \( \vec{0} \in H \)? The zero vector in \( \mathbb{F} \) is the function \( K(x) = 0 \) for all \( x \). It satisfies \( K'(x) = 0 \) for all \( x \) and in particular, \( K'(1) = 0 \).

So \( K \in H \), that is, \( \vec{0} \in H \).

2. Let \( \vec{u} \) and \( \vec{v} \) be in \( H \); we must show that \( \vec{u} + \vec{v} \) is in \( H \).

Now, \( \vec{u} \in H \) means that \( \vec{u} \) is a function \( g \) in \( \mathbb{F} \) for which \( g'(1) = 0 \), and \( \vec{v} \in H \) means that \( \vec{v} \) is a function \( h \) in \( \mathbb{F} \) for which \( h'(1) = 0 \).

Consider \( \vec{u} + \vec{v} \). It is the function \( g + h \). By a theorem in calculus, we know that \( (g + h)'(1) = g'(1) + h'(1) = 0 + 0 = 0 \). Since \( (g + h)'(1) = 0 \), \( g + h \) is in \( H \), that is, \( \vec{u} + \vec{v} \) is in \( H \).

3. Let \( \vec{u} \) be in \( H \) and let \( \alpha \in \mathbb{R} \) be a scalar. We need to show that \( \alpha \vec{u} \in H \).

Again since \( \vec{u} \in H \) we know \( \vec{u} \) is a function \( g \) satisfying \( g'(1) = 0 \).

Consider \( \alpha \vec{u} \). Its the function \( \alpha g \) (that is, \( (\alpha g)(x) = \alpha g(x) \) for all \( x \)).

Now from calculus we know that \( (\alpha g)'(x) = \alpha g'(x) \), so in particular

\[
(\alpha g)'(1) = \alpha g'(1) = \alpha \cdot 0 = 0,
\]

and thus \( \alpha g \in H \), or, \( \alpha \vec{u} \in H \).

3C. Consider now the subset \( J \) of \( \mathbb{F} \) consisting of functions \( f \) that are in \( H \) and which also contain the point \((1,1)\) (one of the functions in the picture has this property). Is \( J \) closed under vector addition? Prove it or give a counterexample.

Consider that function (which I've labeled \( c(x) \) in the figure), it belongs to \( H \) since \( c'(1) = 0 \) and indeed is in \( J \) b/c \( c(1) = 1 \).

But \( (c + c)(1) = c(1) + c(1) = 1 + 1 = 2 \), so \( c + c \) is NOT in \( J \) [so we've supplied a counterexample].

(you don't NEED this, but \( c(x) = (x-1)^2 + 1 \). Check this:
\[
c(1) = (1-1)^2 + 1 = 1 \quad \text{and} \quad c'(x) = 3(x-1)^2(1) = 0 = 3(x-1)
\]
so \( c'(1) = 0 \).

But \( (c + c)(x) = 2 \).}
4. Consider the transformation \( T : \mathbb{P}_3 \to \mathbb{P}_3 \) given by \( T(ax^3 + bx^2 + cx + d) = cx^2 + bx \). So \( T \) "chops off" the cubic and constant terms of \( ax^3 + bx^2 + cx + d \), and swaps the coefficients of the \( x^2 \) and \( x \) terms.

4A. Find two different vectors \( \mathbf{p} \) and \( \mathbf{q} \) in \( \mathbb{P}_3 \) for which \( T(\mathbf{p}) \) and \( T(\mathbf{q}) \) are both equal to \( 4x^2 + 5x \) or explain why this cannot be done.

There are lots of examples:
\[
\alpha \mathbf{p} = x^3 + 5x^2 + 4x + 6 \quad \text{and} \quad \alpha \mathbf{q} = 6x^3 + 5x^2 + 4x + 9. \quad \text{Example}
\]

Then \( T(\alpha \mathbf{p}) = 4x^2 + 5x = T(\alpha \mathbf{q}) \), yet \( \mathbf{p} \neq \mathbf{q} \). [Note: this says that \( T \) is NOT 1-1; see YD below.]

(Indeed \( T(\mathbf{v}) = 4x^2 + 5x \) for any vector \( \mathbf{v} \) of the form \( \mathbf{v} = ax^3 + 5x^2 + 4x + d \) when \( a \) & \( d \) are "free".)

4B. Find two different vectors \( \mathbf{a} \) and \( \mathbf{b} \) in \( \mathbb{P}_3 \) for which \( T(\mathbf{a}) \) and \( T(\mathbf{b}) \) are both equal to \( 7x^3 + 5x^2 \) or explain why this cannot be done.

This is impossible, because \( T(ax^3 + bx^2 + cx + d) = cx^2 + bx \) has no \( x^3 \) term, so \( T(ax^3 + bx^2 + cx + d) \) cannot equal \( 7x^3 + 5x^2 \) no matter the choice of \( a, b, c, d \).

[Note: this means that \( T \) is NOT onto \( \mathbb{R}^3 \) because if \( \mathbf{b} = 7x^3 + 5x^2 \) then \( \mathbf{b} \) is an example of a member of \( \mathbb{R}^3 \) for which \( T(\mathbf{x}) = \mathbf{b} \) has no solution \( \mathbf{x} \) for any \( \mathbf{x} \) in \( \mathbb{R}^3 \).]

4C. Show that for any vectors \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{P}_3 \), the transformation \( T \) satisfies the "\( T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \)" part of the definition of "linear transformation".

Let \( \mathbf{u} \) and \( \mathbf{v} \) be members of \( \mathbb{P}_3 \). So \( \mathbf{u} = ax^3 + bx^2 + cx + d \) for some scalars \( a, b, c, \) and \( d \).

And \( \mathbf{v} = ax^3 + \beta x^2 + \gamma x + \delta \) for some scalars \( a, \beta, \gamma, \) and \( \delta \).

Now, \[
T(\mathbf{u} + \mathbf{v}) = T((ax^3 + bx^2 + cx + d) + (ax^3 + \beta x^2 + \gamma x + \delta)) \]
\[
= T((a + a)x^3 + (b + \beta)x^2 + (c + \gamma)x + (d + \delta)) \]
\[
= (c + \gamma)x^2 + (b + \beta)x.
\]

and \( T(\mathbf{u}) + T(\mathbf{v}) = T(ax^3 + bx^2 + cx + d) + T(ax^3 + \beta x^2 + \gamma x + \delta) \)
\[
= cx^2 + bx + \delta x^2 + \beta x \]
\[
= (c + \gamma)x^2 + (b + \beta)x.
\]

Since the expressions in (1) and (2) are identical, we have shown that \( T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \).

4D. Is \( T \) one-to-one? (Say Yes or No and give a very short reason). \( \text{NO} \) (in 4A we gave a counterexample: \( \mathbf{u} \) and \( \mathbf{v} \) satisfy \( T(\mathbf{u}) = T(\mathbf{v}), \) yet \( \mathbf{u} \neq \mathbf{v} \).)

4E. Is \( T \) onto \( \mathbb{P}_3 \)? (Say Yes or No and give a very short reason). \( \text{NO} \) (there is no \( x \in \mathbb{P}_3 \) satisfying \( T(x) = 7x^3 + 5x^2 \).)
5. Consider the elementary matrices \( P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \) and \( Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \).

5A: Suppose when applied to a matrix \( A \) in the order: first step \( P \), second step \( Q \), the operations turn \( A \) into the identity matrix \( I_3 \). Find \( A, A^{-1} \), and \( A^T \) and the inverse of \( A^T \). Label which is which.

We're told that \( Q \left( P(A) \right) = I_3 \),
or \( QPA = I_3 \),
or \( (QP)A = I_3 \). 
\( QP \) is \( A^{-1} \). But \( QP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \).

\( \Rightarrow \quad A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \). 

It's safest (?) to find \( A = (A^{-1})^{-1} \) via calculator; \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1 \end{bmatrix} \).

Next, \( A^T = \begin{bmatrix} 1 & 0 & 5/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \), and finally, \( (A^T)^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \).

5B: What are the inverses of \( P \) and \( Q \)? (Label them).

\( P \) represents "multiply row 3 by 3". The inverse of this operation is "divide row 3 by 3". \( \Rightarrow \quad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \).

\( Q \) represents "\( r_2 \leftarrow r_2 - 5r_1 \)". The inverse would be "\( r_2 \leftarrow r_2 + 5r_1 \)"
\( \Rightarrow \quad Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \).

5C: Bonus: We could apply the matrix \( P \) four-times in a row. What single elementary matrix accomplishes this same task?

"Doing" \( P \) four times would mean multiplying row 3 by 3, then again by 3, one more time by 3, and then a fourth time by 3.

Since \( 3 \times 3 \times 3 \times 3 = 81 \), this is accomplished in one step using \( P^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 81 \end{bmatrix} \).

5D: Bonus: We could apply the matrix \( Q \) four-times in a row. What single elementary matrix accomplishes this same task?

Here we'd be adding \((-5)\) copies of row one to row three once, twice, three, and four times.

to add \((-5)\) times \( i \) to add \(-20 \) once,
\( \Rightarrow \quad \text{the answer is } Q^4 \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -20 & 0 & 1 \end{bmatrix} \right) \).

(\text{It's NOT } Q \text{ verify } Q^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -20 & 0 & 1 \end{bmatrix} \text{ is NOT even an elementary matrix!})

\( Q^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -20 & 0 & 1 \end{bmatrix} \text{ is NOT even an elementary matrix!} \)

\( \text{I just want to emphasize that it's WRONG to say } PQA = I_3 \text{ and so } PQ = A^{-1}! \)

The order is WRONG, and \( PQ \) is \( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -15 & 0 & 3 \end{bmatrix} \), which is \text{ NOT } the \( A^{-1} \) we found in 5A.

The other three answers are similarly incorrect.