**1.** Consider the following polynomials in $\mathbb{P}_2$:

$\mathbf{p}_1 = 10x^2 + 5x + 1,\qquad \mathbf{p}_2 = 9x^2 + 4x,\qquad \mathbf{p}_3 = 11x^2 + x - 7,\qquad$ and $\qquad \mathbf{p}_4 = x^2 + x + 1.$

Let $H = \text{Span}\{\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3,\ \mathbf{p}_4\}$.

1A. What conditions must $A$, $B$ and $C$ satisfy for $\mathbf{b} = Ax^2 + Bx + C$ to be in $H$? *Show all matrices used* in determining your answer.

1B. Verify that $\mathbf{d} = 24x^2 + 4x - 12$ satisfies the conditions in 1A.

1C. Find all possible ways to express $\mathbf{d}$ as a linear combination $\alpha\mathbf{p}_1 + \beta\mathbf{p}_2 + \gamma\mathbf{p}_3 + \delta\mathbf{p}_4$ of $\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3,$ and $\mathbf{p}_4$.

1D. (Very short yes-or-no answers) Let $S = \{\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3,\ \mathbf{p}_4\}$.

Is $S$ a linearly independent set?

Does $S$ span (all of) $\mathbb{P}_2$?

Does $S$ span (all of) $H$?

Is $S$ a basis for $H$?

**2.** Let $M = \begin{bmatrix} 4 & -6 & 5 & 11 & 5 \\ -2 & 3 & -7 & -1 & 2 \\ 2 & -3 & 4 & 4 & 1 \end{bmatrix}$, let $R$ be the matrix RREF($M$), let $\mathbf{b} = \begin{bmatrix} 2 \\ 8 \\ -2 \end{bmatrix}$ and $\mathbf{n} = \begin{bmatrix} -15 \\ 8 \\ 9 \\ 3 \\ 6 \end{bmatrix}$.

Fact: the RREF of $[M \mid \mathbf{b}]$ is $\left[\begin{array}{ccccc|c} 1 & -3/2 & 0 & 4 & 5/2 & 3 \\ 0 & 0 & 1 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

2A. Does Col $M$ = Col $R$? Explain. If they are not equal, find a vector in one that's not in the other.

2B. Does Nul $M$ = Nul $R$? Explain. If they are not equal, find a vector in one that's not in the other.

2C. Find a basis for Col $M$.

2D. Find a basis for Col $R$.

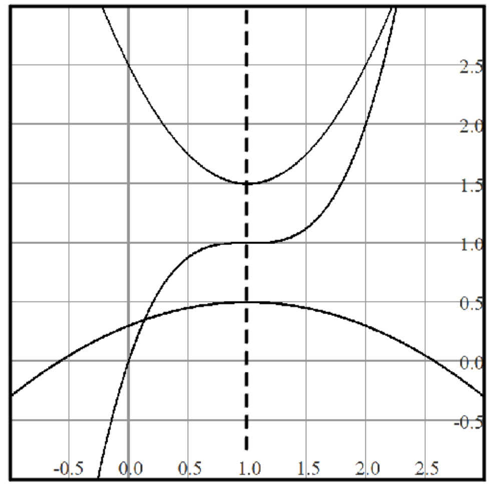
2E. Show how to express $\mathbf{b}$ as a linear combination of the basis vectors in 2C.

2F. Find a basis for Nul $M$.

2G. Find a basis for Nul $R$.

2H. It's a fact that $\mathbf{n} \in$ Nul $M$. Express $\mathbf{n}$ as a linear combination of your basis vectors in 2F.

**3.** Consider the vector space $\mathbb{F}$ of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. Let $H$ be the subset of functions in $\mathbb{F}$ that have horizontal tangents when they intersect the vertical line $x = 1$. Three typical members of $H$ are drawn in the figure to the right.




3A. For any member of $f$ in $H$, what is $f'(1)$?

3B. Give a well-written proof that $H$ is a subspace of $\mathbb{F}$.

3C. Consider now the subset $J$ of $\mathbb{F}$ consisting of functions $f$ that are in $H$ and which also contain the point $(1, 1)$ (one of the functions in the picture has this property). Is $J$ closed under vector addition? Prove it or give a counterexample.

**4.** Consider the transformation $T : \mathbb{P}_3 \to \mathbb{P}_3$ given by $T(ax^3 + bx^2 + cx + d) = cx^2 + bx$. So $T$ "chops off" the cubic and constant terms of $ax^3 + bx^2 + cx + d$, and swaps the coefficients of the $x^2$ and $x$ terms.

4A. Find two different vectors $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{P}_3$ for which $T(\mathbf{p})$ and $T(\mathbf{q})$ are both equal to $4x^2 + 5x$ or explain why this cannot be done.

4B. Find two different vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{P}_3$ for which $T(\mathbf{a})$ and $T(\mathbf{b})$ are both equal to $7x^3 + 5x^2$ or explain why this cannot be done.

4C. Show that for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{P}_3$, the transformation $T$ satisfies the "$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$" part of the definition of "linear transformation".

4D. Is $T$ one-to-one? (Say Yes or No and give a very short reason).

4E. Is $T$ onto $\mathbb{P}_3$? (Say Yes or No and give a very short reason).

**5.** Consider the elementary matrices $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$.

5A: Suppose when applied to a matrix $A$ in the order: first step $P$, second step $Q$, the operations turn $A$ into the identity matrix $I_3$. Find $A$, $A^{-1}$, and $A^T$ and the inverse of $A^T$. Label which is which.

5B: What are the inverses of $P$ and $Q$? (Label them).

5C: Bonus: We could apply the matrix $P$ four-times in a row. What single elementary matrix accomplishes this same task?

5D: Bonus: We could apply the matrix $Q$ four-times in a row. What single elementary matrix accomplishes this same task?