## MATH106A,B CALCULUS II - PROF. P. WONG

EXAM II - MARCH 7, 2008

**NAME:**

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

*Advice*: DON'T spend too much time on a single problem.

| **Problems** | **Maximum Score** | **Your Score** |
|---|---|---|
| 1. | 20 | |
| 2. | 20 | |
| 3. | 20 | |
| 4. | 20 | |
| 5. | 20 | |
| **Total** | 100 | |

**1.**(10 pts.)(a) Evaluate the indefinite integral

$$\int x^2 \ln(2x)\ dx.$$

**Let** $u = \ln(2x)$ **and** $dv = x^2 dx$**. Thus,** $du = \frac{1}{2x} \cdot 2dx = \frac{dx}{x}$ **and** $v = \frac{x^3}{3}$**. Using the technique of integration by parts,**

$$\begin{aligned} \int x^2 \ln(2x)\ dx &= (\ln(2x)) \cdot \left(\frac{x^3}{3}\right) - \int \frac{x^3}{3} \cdot \frac{dx}{x} \\ &= \frac{x^3}{3}\ln(2x) - \frac{1}{3}\int x^2\ dx \\ &= \frac{x^3}{3}\ln(2x) - \frac{x^3}{9} + C. \end{aligned}$$

(10 pts.)(b) Evaluate the indefinite integral

$$\int \frac{x^3+3}{(x+1)x}\ dx.$$

**Applying long division, we obtain**

$$\frac{x^3+3}{(x+1)x} = (x-1) + \frac{x+3}{(x+1)x}.$$

**Now, we use partial fractions for** $\frac{x+3}{(x+1)x}$**. First, we write**

$$\frac{x+3}{(x+1)x} = \frac{A}{(x+1)} + \frac{B}{x}.$$

**Therefore,** $Ax + B(x+1) \equiv x+3$**. It follows that** $A+B=1$ **and** $B=3$ **and so** $A=-2$**. It follows that**

$$\begin{aligned} \int \frac{x^3+3}{(x+1)x}\ dx &= \int (x-1)\ dx + \int \frac{x+3}{(x+1)x}\ dx \\ &= \int (x-1)\ dx + \int \frac{-2}{(x+1)}\ dx + \int \frac{3}{x}\ dx \\ &= \frac{x^2}{2} - x - 2\ln|x+1| + 3\ln|x| + C. \end{aligned}$$

**2.**(10 pts.) Find the indefinite integral

$$\int \frac{1}{x^2\sqrt{1+x^2}}\ dx.$$

**Let** $x = \tan\theta$**. Then,** $dx = \sec^2\theta d\theta, \sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sec\theta$**. Thus,**

$$\int \frac{1}{x^2\sqrt{1+x^2}}\ dx = \int \frac{\sec^2\theta\ d\theta}{\tan^2\theta\sec\theta} = \int \frac{\sec\theta}{\tan^2\theta}\ d\theta$$

$$= \int \frac{\frac{1}{\cos\theta}}{\left(\frac{\sin^2\theta}{\cos^2\theta}\right)}\ d\theta = \int \frac{\cos\theta}{\sin^2\theta}\ d\theta.$$

**Now, let** $u = \sin\theta$ **so that** $du = \cos\theta d\theta$**. Then,**

$$\int \frac{\cos\theta}{\sin^2\theta}\ d\theta = \int \frac{du}{u^2} = -u^{-1} + C = -\frac{1}{\sin\theta} + C.$$

**Since** $x = \tan\theta$**, it follows that** $\sin\theta = \frac{x}{\sqrt{1+x^2}}$**. Hence,**

$$\int \frac{1}{x^2\sqrt{1+x^2}}\ dx = -\frac{\sqrt{1+x^2}}{x} + C.$$

(10 pts.)(b) Find the indefinite integral

$$\int \frac{1}{4x^2+4x+10}\ dx.$$

**The technique of completing the square yields**

$$4x^2+4x+10 = (4x^2+4x+1)+9 = (2x+1)^2+3^2$$

**which is of the form "**$u^2+a^2$**". Now, we let** $(2x+1) = 3\tan\theta$ **so that** $2\ dx = 3\sec^2\theta\ d\theta$ **or** $dx = \frac{3}{2}\sec^2\theta\ d\theta$**. Then,**

$$4x^2+4x+10 = 3^2\tan^2\theta + 3^2 = 3^2(\tan^2\theta+1) = 3^2\sec^2\theta.$$

**It follows that**

$$\int \frac{1}{4x^2+4x+10}\ dx = \int \frac{\frac{3}{2}\sec^2\theta\ d\theta}{3^2\sec^2\theta} = \frac{1}{6}\int\ d\theta$$

$$= \frac{1}{6}\theta + C = \frac{1}{6}\arctan\left(\frac{2x+1}{3}\right) + C.$$

**3.** (10 pts.)(a) Let $f(x) = \arctan x$. Write down the third-degree Maclaurin polynomial $M_3(x)$ for $f$.

**Since** $f(x) = \arctan x$**, we have** $f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}, f''(x) = -(1+x^2)^{-2} \cdot 2x$ **and** $f'''(x) = [2(1+x^2)^{-3} \cdot 2x] \cdot 2x + [-(1+x^2)^{-2}] \cdot 2$**. Now,**

$$f(0) = 0, f'(0) = 1, f''(0) = 0 \text{ and } f'''(0) = -2.$$

**It follows that**

$$\begin{aligned} M_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ &= x - \frac{2}{3!}x^3 = x - \frac{x^3}{3}. \end{aligned}$$

(10 pts.)(b) Let $g(x) = \sqrt{x}$. Find the third-degree Taylor polynomial $P_3(x)$ for $g(x)$ centered at $x_0 = 4$.

**Since** $g(x) = \sqrt{x} = x^{1/2}$**, we have** $g'(x) = \frac{1}{2}x^{-1/2}, g''(x) = -\frac{1}{4}x^{-3/2}$ **and** $g'''(x) = \frac{3}{8}x^{-5/2}$**. Now**

$$g(4) = 2, g'(4) = \frac{1}{4}, g''(4) = -\frac{1}{32} \text{ and } g'''(x) = \frac{3}{8} \cdot \frac{1}{2^5} = \frac{3}{256}.$$

**It follows that**

$$\begin{aligned} P_3(x) &= g(4) + g'(4)(x-4) + \frac{g''(4)}{2!}(x-4)^2 + \frac{g'''(4)}{3!}(x-4)^3 \\ &= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3. \end{aligned}$$

**4.**(10 pts.)(a) Let $f(x) = \cos(2x)$. What is the maximum possible error, according to Taylor's theorem, committed by using the third-degree Maclaurin polynomial $M_3(x)$ to estimate $f(x)$ for $-2 \leq x \leq 2$?

**Since** $f(x) = \cos(2x)$**, then** $f'(x) = -2\sin(2x), f''(x) = -4\cos(2x), f'''(x) = 8\sin(2x)$**, and** $f^{(4)}(x) = 16\cos(2x)$**. Note that** $|f^{(4)}(x)| \leq 16$ **so we can choose** $K_4 = 16$**. Over the interval** $[-2, 2]$**,** $|x| \leq 2$**. Therefore, Taylor's theorem asserts that the maximum error committed by** $M_3(x)$ **is given by**

$$\frac{K_4|x|^4}{4!} \leq \frac{16(2)^4}{4!} = \frac{32}{3}.$$

(10 pts.)(b) Let

$$h(x) = \begin{cases} k\sin x, & \text{if } 0 \leq x \leq \pi; \\ 0, & \text{otherwise.} \end{cases}$$

Find $k$ for which $h(x)$ is a probability density function.

**For** $h(x)$ **to be a probability density function,** $h(x) \geq 0$ **for all** $x$**. Since** $\sin x \geq 0$ **for** $0 \leq x \leq \pi$**, this implies that** $k \geq 0$**. In addition,** $\int_{-\infty}^{\infty} h(x)\ dx$ **must be equal to** $1$**. It follows that**

$$1 = \int_{-\infty}^{\infty} h(x)\ dx = \int_0^{\pi} k\sin x\ dx = -k\cos x\ \Big|_0^{\pi} = -k(-1) - (-k).$$

**This means that** $2k = 1$ **or** $k = \frac{1}{2}$**.**

**5.** Determine whether each of the following improper integrals converges or diverges. Justify your answers.

(10 pts.)(a)

$$\int_0^\infty \frac{1}{\sqrt{1+x^4}}\ dx$$

[Hint: compare this integral with another improper integral]

**First,**

$$\int_0^\infty \frac{1}{\sqrt{1+x^4}}\ dx = \int_0^1 \frac{1}{\sqrt{1+x^4}}\ dx + \int_1^\infty \frac{1}{\sqrt{1+x^4}}\ dx.$$

**Note that** $\int_0^1 \frac{1}{\sqrt{1+x^4}}\ dx$ **is finite and**

$$\int_1^\infty \frac{1}{\sqrt{1+x^4}}\ dx \le \int_1^\infty \frac{1}{\sqrt{x^4}}\ dx = \int_1^\infty \frac{1}{x^2}\ dx < \infty.$$

**It follows that the improper integral** $\int_0^\infty \frac{1}{\sqrt{1+x^4}}\ dx$ **is finite and hence converges.**

(10 pts.)(b)

$$\int_0^1 \frac{1}{\sqrt{1-x}}\ dx$$

**First of all, this integral is improper so that**

$$\int_0^1 \frac{1}{\sqrt{1-x}}\ dx = \lim_{b\to 1^-} \int_0^b \frac{1}{\sqrt{1-x}}\ dx.$$

**Note that** $\int \frac{1}{\sqrt{1-x}}\ dx = \int \frac{-du}{\sqrt{u}}$ **where** $u = 1-x$ **with** $dx = -du$**. Thus,**

$$\int \frac{1}{\sqrt{1-x}}\ dx = \frac{-u^{1/2}}{1/2} = -2\sqrt{u} + C = -2\sqrt{1-x} + C.$$

**Thus,**

$$\begin{aligned}\lim_{b\to 1^-} \int_0^b \frac{1}{\sqrt{1-x}}\ dx &= \lim_{b\to 1^-} -2\sqrt{1-x}\ \Big|_0^b \\ &= \lim_{b\to 1^-} -2\sqrt{1-b} + 2 \\ &= 0 + 2 = 2 < \infty.\end{aligned}$$

**Hence the improper integral converges.**