1. (10 points) Evaluate $\int \frac{dx}{\sqrt{16+x^2}}$.

a^2+x^2 form $\rightarrow$ $x = a\tan t$.

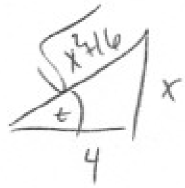
$a=4$. Let $x = 4\tan t$, so $dx = 4\sec^2 t\,dt$.

$$\int \frac{dx}{\sqrt{16+x^2}} = \int \frac{4\sec^2 t\,dt}{\sqrt{16+16\tan^2 t}} = \int \frac{4\sec^2 t\,dt}{\sqrt{16\sec^2 t}} = \int \frac{4\sec^2 t\,dt}{4\sec t\,dt}$$

$$= \int \sec t\,dt \overset{\text{by formula \#1}}{=} \ln|\sec t + \tan t| + C$$

$$\boxed{= \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C.}$$

$x = 4\tan t$,
so $\tan t = \frac{x}{4}$



So $\sec t = \frac{hyp}{adj} = \frac{\sqrt{x^2+16}}{4}$

2. (10 points) Evaluate $\int xe^{3x}\,dx$. Parts.

Let $u = x$ $\qquad dv = e^{3x}dx$

So $du = dx$ $\qquad v = \frac{1}{3}e^{3x}$

$$\int xe^{3x}dx = x\cdot\frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x}dx$$

$$= \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx$$

$$\boxed{= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C.}$$

3. (20 points) Let $f(x) = \sqrt{x}$.

(a) Find $P_2(x)$, the degree-2 Taylor polynomial for $f(x)$ centered at $x = 4$.

$f(x) = \sqrt{x} = x^{1/2} \rightarrow f(4) = \sqrt{4} = 2$

$f'(x) = \frac{1}{2} x^{-1/2} \rightarrow f'(4) = \frac{1}{2}(4)^{-1/2} = \frac{1}{4}$

$f''(x) = -\frac{1}{4} x^{-3/2} \rightarrow f''(4) = -\frac{1}{4} 4^{-3/2} = -\frac{1}{32}$.

$P_2(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2$

$\boxed{P_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2}$

(b) Use your polynomial to approximate $\sqrt{6}$.

$P_2(x) \approx \sqrt{x}$ for x-values near 4.

So $P_2(6) \approx \sqrt{6}$.

$P_2(6) = 2 + \frac{1}{4}(6-4) - \frac{1}{64}(6-4)^2$

$= 2 + \frac{1}{4}(2) - \frac{1}{64}(4) = \boxed{\frac{39}{16}}$

(c) What does Taylor's Theorem imply about the maximum approximation error committed by $P_2(x)$ on the interval $[4, 6]$?

If $|f'''(x)| \le K_3$ on $[4,6]$, then $|f(x) - P_2(x)| \le \frac{K_3 |x - x_0|^3}{3!}$.

$f'''(x) = \frac{3}{8} x^{-5/2}$. Max value on $[4,6]$ occurs at $x = 4$.

$f'''(4) = \frac{3}{256} \leftarrow$ Let $K_3 = \frac{3}{256}$.

Then $|f(x) - P_2(x)| \le \frac{K_3}{3!}|x-4|^3 = \frac{3/256}{6}|x-4|^3$

$= \frac{1}{512}|x-4|^3$

$\le \frac{1}{512}|6-4|^3 = \frac{1}{64}$.

$\boxed{\text{Max possible error on } [4,6] \text{ is } \frac{1}{64}.}$

4. (10 points) A certain company sells light bulbs. Let X be the length of time, in days, that it takes for a randomly chosen light bulb to burn out. The probability density function of X is

$$f(x) = \begin{cases} \frac{k}{(x+1)^3} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

for some positive constant k. Find the value of k that makes $f(x)$ a probability density function.

① $f(x)$ defined everywhere. ✓

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$. (Details below)

③ $f(x) \geq 0$. ✓

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} \frac{k}{(x+1)^3}\,dx$$

Let $u = 1+x$
$du = dx$.

$$= \int_0^{\infty} \frac{k}{(1+x)^3}\,dx = \lim_{t\to\infty} k \int_0^t (1+x)^{-3}\,dx.$$

$$= \lim_{t\to\infty} k \int_1^{t+1} u^{-3}\,du$$

$$= \lim_{t\to\infty} k \left[\frac{u^{-2}}{-2} \right]_1^{t+1}$$

$$= \lim_{t\to\infty} k \left(\frac{(t+1)^{-2}}{-2} - \frac{(1)^{-2}}{-2} \right)$$

$$= \lim_{t\to\infty} k \left(\frac{-1}{2(t+1)^2} + \frac{1}{2} \right)$$

$$= k\left(0 + \frac{1}{2}\right)$$

$$= \frac{k}{2}.$$

Need $1 = \frac{k}{2}$, so $\boxed{k = 2.}$

5. (10 points) Use a comparison to determine whether $\int_1^\infty \frac{1}{\sqrt{x} + x^4}dx$ converges or diverges.

A comparison to $\int_1^\infty \frac{1}{\sqrt{x}}dx$ doesn't help because

$\int_1^\infty \frac{1}{\sqrt{x}}dx > \int_1^\infty \frac{1}{\sqrt{x}+x^4}dx$, and $\int_1^\infty \frac{1}{\sqrt{x}}dx$ diverges

However, comparing to $\int_1^\infty \frac{1}{x^4}dx$, we see that

Since $\sqrt{x}+x^4 > x^4$, then $\frac{1}{\sqrt{x}+x^4} < \frac{1}{x^4}$, so

$$\int_1^\infty \frac{1}{\sqrt{x}+x^4}dx < \int_1^\infty \frac{1}{x^4}dx.$$

of form $\int_1^\infty \frac{1}{x^p}dx$ with $p=4$.
This converges for $p>1$.

So $\int_1^\infty \frac{1}{\sqrt{x}+x^4}dx$ is a positive integral bounded above by a convergent integral, hence $\int_1^\infty \frac{1}{\sqrt{x}+x^4}dx$ converges, too.

Potentially Useful Integral Formulæ

1. $\int \sec(x)\ dx = \ln|\sec x + \tan x| + C.$

2. $\int \sin^n(x)\ dx = -\frac{\sin^{n-1}(x)cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)dx, \text{ for } n \neq 0.$

3. $\int \cos^n(x)\ dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)dx, \text{ for } n \neq 0.$

4. $\int \sec^n(x)\ dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx, \text{ for } n \neq 1.$