

MATH 205_A Exam 2

March 7, 2008

Name

Suggested solutions

1
2
3
4
5
6
TOTAL

- ① read the questions
- ② answer in the provided spaces
- ③ Show all work
- ④ BE NEAT

Good Luck!



Here are two facts you may find useful:

For problem ONE:

$$\begin{bmatrix} 4 & 2 & 6 & 14 & 24 \\ -1 & 5 & 4 & 10 & 5 \\ 2 & 2 & 4 & 24 & 14 \end{bmatrix} \text{ is row equivalent to } \begin{bmatrix} 1 & 0 & 1 & 0 & 5 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For problem TWO:

$$\begin{bmatrix} -1 & 5 & 18 & | & 1 & 0 & 0 & 0 \\ -2 & 5 & 14 & | & 0 & 1 & 0 & 0 \\ 1 & -3 & -9 & | & 0 & 0 & 1 & 0 \\ -2 & 3 & 10 & | & 0 & 0 & 0 & 1 \end{bmatrix} \text{ is row equivalent to } \begin{bmatrix} 1 & 0 & 0 & | & 0 & -3/4 & -2 & -3/4 \\ 0 & 1 & 0 & | & 0 & 2 & 2 & -1 \\ 0 & 0 & 1 & | & 0 & -3/4 & -1 & 1/4 \\ 0 & 0 & 0 & | & 1 & 11/4 & 6 & -1/4 \end{bmatrix}$$

1. Suppose $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ is a linear transformation and it's given that

$$T\left(\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ 3 \\ 8 \\ -11 \end{bmatrix}, \quad T\left(\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -14 \\ 29 \\ 38 \\ 27 \end{bmatrix}, \quad \text{and} \quad T\left(\begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -6 \\ 32 \\ 46 \\ 16 \end{bmatrix}.$$

1A. Express $x_1 = \begin{bmatrix} 14 \\ 10 \\ 24 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}$, or explain why you can't.

you CAN'T: the row reduction of $\left[\begin{array}{ccc|c} 4 & 2 & 6 & 14 \\ -1 & 5 & 4 & 10 \\ 2 & 2 & 4 & 24 \end{array} \right]$ to $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$ ← (see info on page 0!)

shows the system $x_1 \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \\ 24 \end{bmatrix}$ is inconsistent.

1B. Find $T(x_1)$ or explain why you can't, based on what you know about T .

Since we only know the values of $T\left(\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}\right)$, $T\left(\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}\right)$, $T\left(\begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}\right)$,

and are unable to write $\begin{bmatrix} 14 \\ 10 \\ 24 \end{bmatrix}$ as a L.C. of $\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}$,

we can't find $T\left(\begin{bmatrix} 14 \\ 10 \\ 24 \end{bmatrix}\right)$.

1C. Express $x_2 = \begin{bmatrix} 24 \\ 5 \\ 14 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}$, or explain why you can't.

you CAN; row reduction of $\left[\begin{array}{ccc|c} 4 & 2 & 6 & 24 \\ -1 & 5 & 4 & 5 \\ 2 & 2 & 4 & 14 \end{array} \right]$ to $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$ ← (see info on page 0)

shows the soln's of $x_1 \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 24 \\ 5 \\ 14 \end{bmatrix}$ are $\begin{cases} x_1 = 5 - x_3 \\ x_2 = 2 - x_3 \\ x_3 = \text{free} \end{cases}$

In particular, let $x_3 = 0$, then $5 \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 24 \\ 5 \\ 14 \end{bmatrix}$, ...

1D. Find $T(x_2)$ or explain why you can't, based on what you know about T .

$$\dots, \text{ so } T\left(\begin{bmatrix} 24 \\ 5 \\ 14 \end{bmatrix}\right) = 5 T\left(\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}\right) + 2 T\left(\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}\right) = 5 \begin{bmatrix} 8 \\ 3 \\ 8 \\ -11 \end{bmatrix} + 2 \begin{bmatrix} -14 \\ 29 \\ 38 \\ 27 \end{bmatrix} = \begin{bmatrix} 12 \\ 73 \\ 116 \\ -1 \end{bmatrix}$$

2. Suppose $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ is a linear transformation with standard matrix $A = \begin{bmatrix} -1 & 5 & 18 \\ -2 & 5 & 14 \\ 1 & -3 & -9 \\ -2 & 3 & 10 \end{bmatrix}$.

2A. Find $T\left(\begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}\right) = A \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

2B. Is T onto \mathbb{R}^4 ? Explain your answer.

No, since solving $A\vec{x} = \vec{b}$ is not always possible as the RREF of A has a row of zeros which means $A\vec{x} = \vec{b}$ could be inconsistent (see the RREF, page 0)

2C. Are there any conditions that a vector \vec{b} must satisfy in order to be in the image of T ? If so, what are they?

in fact, the RREF on page 0 shows

$$\vec{b} \in \text{image of } T \iff b_1 + \frac{11}{4}b_2 + 6b_3 - \frac{1}{4}b_4 = 0$$

2D. Give an example of a vector which is *not* in the image of T , or explain why this cannot be done.

just choose \vec{b} so that the condition in 2C is not met:

for example, $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ or $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ or $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

2E. Is T one-to-one? Explain your answer.

yes. suppose $T(\vec{x}) = \vec{b}$, i.e. $A\vec{x} = \vec{b}$, has a soln' \vec{p} . (so $A\vec{p} = \vec{b}$)
Row reduction of A shows it has no free variables, so there is only the one soln' to $A\vec{x} = \vec{b}$, namely \vec{p} .

2F. Give three different vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, each satisfying $T(\vec{x}) = \vec{0}$, or explain why this cannot be done.

again, this is impossible b/c T is 1-1, so since $T(\vec{0}) = \vec{0}$,

if $T(\vec{v}) = \vec{0}$ also then $\vec{0} = \vec{v}$

(alternatively, the only soln' to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$ since RREF(A) shows A has no free variables)

3. Suppose an economy is modeled with four sectors A , B , C , and D . Suppose that the output of D is evenly divided (consumed) by all four sectors, while A and B each buy 75% of each other's output yet keep none of their own (respectively). Suppose the remainder of A 's output is purchased by C . Suppose that none of C 's output is sold to B and *vice versa*, while A and D each use 40% of C 's output.

3A. Find the exchange table for this economy. You may assume all columns sum to one.

	A	B	C	D	
A	0	$\frac{3}{4}$	$\frac{2}{5}$	$\frac{1}{4}$	A
B	$\frac{3}{4}$	0	0	$\frac{1}{4}$	B
C	$\frac{1}{4}$	0	$\frac{1}{5}$	$\frac{1}{4}$	C
D	0	$\frac{1}{4}$	$\frac{2}{5}$	$\frac{1}{4}$	D

3B. Find the complete set $\{P_A, P_B, P_C, P_D\}$ of equilibrium solutions for this economy. Write down any system of equations and augmented matrices you use in solving this problem.

We must solve the system

$$\begin{cases} P_A = 0P_A + \frac{3}{4}P_B + \frac{2}{5}P_C + \frac{1}{4}P_D \\ P_B = \frac{3}{4}P_A + 0P_B + 0P_C + \frac{1}{4}P_D \\ P_C = \frac{1}{4}P_A + 0P_B + \frac{1}{5}P_C + \frac{1}{4}P_D \\ P_D = 0P_A + \frac{1}{4}P_B + \frac{2}{5}P_C + \frac{1}{4}P_D \end{cases}$$

the corresponding augmented matrix is

$$\left[\begin{array}{cccc|c} -1 & \frac{3}{4} & \frac{2}{5} & \frac{1}{4} & 0 \\ \frac{3}{4} & -1 & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & -\frac{1}{5} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{2}{5} & -\frac{3}{4} & 0 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{5} & 0 \\ 0 & 1 & 0 & -\frac{8}{5} & 0 \\ 0 & 0 & 1 & -\frac{7}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{cases} P_A = \frac{9}{5}P_D \\ P_B = \frac{8}{5}P_D \\ P_C = \frac{7}{8}P_D \\ P_D = \text{free} \end{cases}$$

3C. Suppose P_D is 100 dollars. Rank all four equilibrium prices from least to greatest.

indeed for any value of $P_D > 0$ we have

$$P_C < P_D < P_B < P_A$$

4. Let $M = \begin{bmatrix} 4 & a & 0 \\ 0 & 3 & b \\ 0 & 0 & 1 \end{bmatrix}$.

4A. In terms of a and b , find the inverse of M using the " $[A|I] \sim [I|A^{-1}]$ " algorithm discussed in class and show all your steps.

(as usual, there are many different "paths" to the final RREF form of A . Here's one of them. Rows that change are written in **BOLD**)

$$\begin{bmatrix} 4 & a & 0 & | & 1 & 0 & 0 \\ 0 & 3 & b & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & a/4 & 0 & | & 1/4 & 0 & 0 \\ 0 & 3 & b & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \leftarrow r_1 - \frac{a}{4}r_2} \begin{bmatrix} 1 & 0 & -\frac{ab}{12} & | & 1/4 & -\frac{a}{12} & 0 \\ 0 & 3 & b & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_2 \leftarrow r_2 - \frac{b}{3}r_3} \begin{bmatrix} 1 & 0 & -\frac{ab}{12} & | & 1/4 & -\frac{a}{12} & 0 \\ 0 & 3 & 0 & | & 0 & 1 & -\frac{b}{3} \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \leftarrow r_1 + \frac{ab}{12}r_3} \begin{bmatrix} 1 & 0 & 0 & | & 1/4 & -\frac{a}{12} & \frac{ab}{12} \\ 0 & 3 & 0 & | & 0 & 1 & -\frac{b}{3} \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

4B. Find the determinants of each of the following matrices and write your answers in the boxes.

3M

324

NOTE $\det(M) = 4 \cdot 3 \cdot 1 = 12$ b/c it's upper triangular.

every row of M gets multiplied by 3 (to produce $3M$)
so $\det(3M) = 3 \cdot 3 \cdot 3 \cdot \det(M) = 27 \cdot 12 = 324$

M^3

1728

$$\begin{aligned} \det(M^3) &= \det(M \cdot M \cdot M) \\ &= \det(M) \det(M) \det(M) \\ &= (\det(M))^3 = 12^3 = 1728 \end{aligned}$$

$$\begin{bmatrix} 4 & a & 0 \\ 4 & 3+a & b \\ 8 & 8a & 1 \end{bmatrix}$$

N/A

this has a typo; problem discarded

$$\begin{bmatrix} 4 & a & -1 \\ 0 & 3 & b \\ 8 & 2a+3 & b-2 \end{bmatrix}$$

12

this matrix results from A by doing $r_3 \leftarrow r_3 - r_1$ then $r_3 \leftarrow r_3 + 2r_1$ then $r_3 \leftarrow r_3 + r_2$. None of these change the det from $\det(A)$

$2M + 3I_3$

495

WRONG: " $\det(2M) + \det(3I_3)$ "; there is NO rule like $\det(A+B) = \det(A) + \det(B)$

$$\begin{aligned} \text{BUT! } 2M + 3I_3 &= \begin{bmatrix} 8 & 2a & 0 \\ 0 & 6 & 2b \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 2a & 0 \\ 0 & 9 & 2b \\ 0 & 0 & 5 \end{bmatrix} \text{ which is upper triangular, so det is } 11 \cdot 9 \cdot 5 = 495 \end{aligned}$$

$M^{-1}M^T$

1

$$\begin{aligned} \det(M^{-1} \cdot M^T) &= \det(M^{-1}) \cdot \det(M^T) \\ &= \frac{1}{12} \cdot 12 = 1 \end{aligned}$$

5. Consider the vector space F of functions $f: \mathbb{R} \rightarrow \mathbb{R}$ which are continuous everywhere, as discussed in class. Let H consist of all members of F which are differentiable at $x = 1$ and furthermore, the line tangent to the graph of any member of H is horizontal at $x = 1$.

5A. What does this last condition say that $h'(1)$ equals, for any member $u = h(x)$ which belongs to H ?

$$h'(1) = 0$$

5B. Which, if any, of the following functions belong to H ? Explain!

a) e^{x-1} $\frac{d}{dx}(e^{x-1})$ is e^{x-1} ; evaluate at $x=1$ to get $e^{1-1} = e^0 = 1 \neq 0$,
so $e^{x-1} \notin H$.

b) $p(x) = 2x^2 - 4x + 8$

$\frac{d}{dx}(2x^2 - 4x + 8) = 4x - 4$; so $p'(1) = 4 \cdot 1 - 4 = 4 - 4 = 0$; thus $p(x) \in H$.

5C. PROVE that H is a subspace of F .

Ⓐ $\boxed{\text{is } \vec{0} \in H?}$ Here $\vec{0}$ is the function $f(x) = 0$; since $f'(x) = 0$ for all x 's, $f'(1) = 0$ & so $f \in H$, i.e. $\vec{0} \in H$.

(alternatively: the graph of $\vec{0}$ is $\leftarrow \overset{1}{|} \rightarrow$ which has a slope of 0 at 1)

Ⓑ $\boxed{\text{Let } \vec{u} \& \vec{v} \in H. \text{ Is } \vec{u} + \vec{v} \in H?}$

Since $\vec{u} \in H$, \vec{u} is a fn f satisfying $f'(1) = 0$

" $\vec{v} \in H$, \vec{v} " " g " $g'(1) = 0$.

Now, $(f(x) + g(x))' = f'(x) + g'(x)$ so when $(f(x) + g(x))'$ is evaluated at $x=1$ we get $f'(1) + g'(1) = 0 + 0 = 0$, so $\vec{u} + \vec{v} \in H$.

(note: its incorrect to write " $(f(1) + g(1))' = 0$." but $(f(x) + g(x))'|_{x=1}$ is fine)

Ⓒ $\boxed{\text{Let } \alpha \in \mathbb{R} \& \vec{u} \in H. \text{ Is } \alpha \vec{u} \in H?}$

since $\vec{u} \in H$, \vec{u} is a fn f satisfying $f'(1) = 0$.

$\alpha \vec{u}$ is the function αf , and $(\alpha f)'|_{x=1} = \alpha f'(1) = \alpha \cdot 0 = 0$

so $\alpha \vec{u}$ is also in H .

6. Let P_3 be the vector space of polynomials of degree three-or-less, and let H be all members of P_3 which have a slope of 1 when $x = 1$.

6A. Find three members of H , one of degree one, one of degree two, and one of degree three. CIRCLE your answers

THERE ARE MANY ANSWERS; these 3 are

a)

x

pretty obvious

b)

$\frac{1}{2}x^2$

c)

$\frac{1}{3}x^3$

Here's the technical soln to say, GA: we need a polynomial of the form

$$p(x) = ax^3 + bx^2 + cx + d \quad \text{where } a \neq 0 \text{ \& } p'(1) = 1.$$

$$\text{now, } p'(x) = 3ax^2 + 2bx + c$$

$$\text{so } p'(1) = 3a + 2b + c;$$

$$\text{so } 3a + 2b + c = 1$$

$$3a = 1 - 2b - c$$

$$a = \frac{1}{3} - \frac{2}{3}b - \frac{c}{3} \quad \text{where } b \text{ \& } c \text{ are free (as long as } a \neq 0)$$

$$\text{for example: } b = c = 0 \Rightarrow a = \frac{1}{3} \dots$$

6B. Show that H is not closed under vector addition.

let \vec{u} be x and \vec{v} be $\frac{1}{2}x^2$,

then \vec{u} \& \vec{v} are both in H .

Now, $\vec{u} + \vec{v}$ is $(x + \frac{1}{2}x^2)$, and the derivative of this is $1 + x$, which is 2, NOT 1, when $x = 1$. $\therefore \vec{u} + \vec{v} \notin H$, and so

H is not closed under vector addition.