# MATH 205A Exam 2
March 7, 2008

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**Total**

- Read the questions
- Answer in the provided spaces
- Show all work
- BE NEAT

Good luck!
Here are two facts you may find useful:

For problem ONE:
\[
\begin{bmatrix}
4 & 2 & 6 & 14 & 24 \\
-1 & 5 & 4 & 10 & 5 \\
2 & 2 & 4 & 24 & 14
\end{bmatrix}
\text{ is row equivalent to }
\begin{bmatrix}
1 & 0 & 1 & 0 & 5 \\
0 & 1 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

For problem TWO:
\[
\begin{bmatrix}
-1 & 5 & 18 & 1 & 0 & 0 & 0 \\
-2 & 5 & 14 & 0 & 1 & 0 & 0 \\
1 & -3 & -9 & 0 & 0 & 1 & 0 \\
-2 & 3 & 10 & 0 & 0 & 0 & 1
\end{bmatrix}
\text{ is row equivalent to }
\begin{bmatrix}
1 & 0 & 0 & 0 & -3/4 & -2 & -3/4 \\
0 & 1 & 0 & 0 & 2 & 2 & -1 \\
0 & 0 & 1 & 0 & -3/4 & -1 & 1/4 \\
0 & 0 & 0 & 1 & 11/4 & 6 & -1/4
\end{bmatrix}
\]
1. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation and it's given that

\[
T \left( \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} -14 \\ 29 \\ 38 \\ 27 \end{bmatrix}, \quad T \left( \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} -6 \\ 32 \\ 46 \\ 16 \end{bmatrix}.
\]

1A. Express $x_1 = \begin{bmatrix} 14 \\ 10 \\ 24 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix}$, or explain why you can't.

You can't; the row reduction of $\left[ \begin{array}{ccc|c} -1 & 2 & 6 & 14 \\ 2 & 2 & 4 & 10 \\ \end{array} \right]$ to $\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \end{array} \right]$ (on page 0) shows the system $x_1 \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \\ 24 \end{bmatrix}$ is inconsistent.

1B. Find $T(x_1)$ or explain why you can't, based on what you know about $T$.

Since we only know the values of $T \left( \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right)$, $T \left( \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} \right)$, and are unable to write $\begin{bmatrix} 14 \\ 10 \\ 24 \end{bmatrix}$ as a LC of $\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix}$, we can't find $T \left( \begin{bmatrix} 14 \\ 10 \\ 24 \end{bmatrix} \right)$.

1C. Express $x_2 = \begin{bmatrix} 24 \\ 5 \\ 14 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix}$, or explain why you can't.

You can; row reduction of $\left[ \begin{array}{ccc|c} 4 & 2 & 6 & 24 \\ -1 & 5 & 4 & 14 \\ \end{array} \right]$ to $\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ \end{array} \right]$ (see info on page 0) shows the solns of $x_1 \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 24 \\ 5 \\ 14 \end{bmatrix}$ are $x_1 = 5 - x_3$, $x_2 = 2 - x_3$, $x_3 = \text{free}$.

In particular, let $x_2 = 0$, then $5 \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 24 \\ 5 \\ 14 \end{bmatrix}$...

1D. Find $T(x_2)$ or explain why you can't, based on what you know about $T$.

...so $T \left( \begin{bmatrix} 24 \\ 5 \\ 14 \end{bmatrix} \right) = 5 T \left( \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right) + 2 T \left( \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} \right) = 5 \begin{bmatrix} 8 \\ 3 \\ -11 \\ 27 \end{bmatrix} + 2 \begin{bmatrix} -14 \\ 29 \\ 38 \\ 27 \end{bmatrix} = \begin{bmatrix} 12 \\ 73 \\ 116 \\ -1 \end{bmatrix}$.
2. Suppose \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \) is a linear transformation with standard matrix \( A = \begin{bmatrix} -1 & 5 & 18 \\ -2 & 5 & 14 \\ 1 & -3 & -9 \\ -2 & 3 & 10 \end{bmatrix} \).

2A. Find \( T \left( \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right) = A \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix} \).

2B. Is \( T \) onto \( \mathbb{R}^4 \)? Explain your answer.

No, since solving \( A\mathbf{x} = \mathbf{b} \) is not always possible as the RREF of \( A \) has a row of zeros, which means \( A\mathbf{x} = \mathbf{b} \) could be inconsistent (see the RREF on page 0).

2C. Are there any conditions that a vector \( \mathbf{b} \) must satisfy in order to be in the image of \( T \)? If so, what are they?

In fact, the RREF on page 0 shows

\( \mathbf{b} \in \text{image of } T \iff b_1 + \frac{11}{9} b_2 + 6 b_3 - \frac{1}{3} b_4 = 0 \).

2D. Give an example of a vector which is not in the image of \( T \), or explain why this cannot be done.

Just choose \( \mathbf{b} \) so that the condition in 2C is not met:

for example, \( \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \) or \( \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \) or \( \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \).

2E. Is \( T \) one-to-one? Explain your answer.

Yes. Suppose \( T(\mathbf{x}) = \mathbf{b} \), i.e. \( A\mathbf{x} = \mathbf{b} \), has a soln \( \hat{\mathbf{x}} \). (so \( A\hat{\mathbf{x}} = \mathbf{b} \))

Row reduction of \( A \) shows it has no free variables, so there is only the one soln to \( A\mathbf{x} = \mathbf{b} \), namely \( \hat{\mathbf{x}} \).

2F. Give three different vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \), each satisfying \( T(\mathbf{x}) = 0 \), or explain why this cannot be done.

Again, this is impossible b/c \( T \) is 1-1, so since \( T(\mathbf{0}) = \mathbf{0} \),

if \( T(\mathbf{v}) = \mathbf{0} \) also then \( \mathbf{0} = \mathbf{v} \)

(alternatively, the only soln to \( A\mathbf{x} = \mathbf{0} \) is \( \mathbf{x} = \mathbf{0} \) since RREF(\( A \)) shows \( A \) has no free variables).
3. Suppose an economy is modeled with four sectors A, B, C, and D. Suppose that the output of D is evenly divided (consumed) by all four sectors, while A and B each buy 75% of each other’s output yet keep none of their own (respectively). Suppose the remainder of A’s output is purchased by C. Suppose that none of C’s output is sold to B and vice versa, while A and D each use 40% of C’s output.

3A. Find the exchange table for this economy. You may assume all columns sum to one.

\[
\begin{array}{cccc}
A & B & C & D \\
0 & \frac{3}{4} & \frac{2}{5} & \frac{1}{4} \\
\frac{3}{4} & 0 & 0 & \frac{1}{4} \\
\frac{1}{4} & 0 & \frac{1}{5} & \frac{1}{4} \\
0 & \frac{1}{4} & \frac{2}{5} & \frac{1}{4} \\
\end{array}
\]

3B. Find the complete set \(\{P_A, P_B, P_C, P_D\}\) of equilibrium solutions for this economy. Write down any system of equations and augmented matrices you use in solving this problem.

We must solve the system

\[
\begin{align*}
P_A &= OP_A + \frac{3}{4}P_B + \frac{2}{5}P_C + \frac{1}{4}P_B \\
P_B &= \frac{3}{4}P_A + OP_B + OP_C + \frac{1}{4}P_D \\
P_C &= \frac{1}{4}P_A + OP_B + \frac{1}{5}P_C + \frac{1}{4}P_D \\
P_D &= OP_A + \frac{1}{4}P_B + \frac{3}{5}P_C + \frac{1}{4}P_D
\end{align*}
\]

The corresponding augmented matrix is

\[
\begin{bmatrix}
-1 & \frac{3}{4} & \frac{2}{5} & \frac{1}{4} & 0 \\
\frac{3}{4} & -1 & 0 & \frac{1}{4} & 0 \\
\frac{1}{4} & 0 & -\frac{1}{5} & \frac{1}{4} & 0 \\
0 & \frac{1}{4} & \frac{2}{5} & -\frac{3}{4} & 0 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 & -\frac{1}{5} & 0 \\
0 & 1 & 0 & -\frac{8}{5} & 0 \\
0 & 0 & 1 & -\frac{7}{8} & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\Rightarrow
\begin{align*}
P_A &= \frac{9}{5}P_D \\
P_B &= \frac{8}{5}P_D \\
P_C &= \frac{7}{8}P_D \\
P_D &= \text{free}
\end{align*}
\]

3C. Suppose \(P_D\) is 100 dollars. Rank all four equilibrium prices from least to greatest.

Indeed for any value of \(P_D > 0\) we have

\[
P_C < P_D < P_B < P_A
\]
4. Let $M = \begin{bmatrix} 4 & a & 0 \\ 0 & 3 & b \\ 0 & 0 & 1 \end{bmatrix}$.

4A. In terms of $a$ and $b$, find the inverse of $M$ using the $"[A|I] \sim [I|A^{-1}]"$ algorithm discussed in class and show all your steps.

(as usual, there are many different "paths" to the final RREF form of $A$. Here's one of them. Pairs that change are written in **bold**)

\[
\begin{bmatrix} 4 & a & 0 \\ 0 & 3 & b \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & a/4 & 0 \\ 0 & 1 & \frac{b}{3} \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{-ab}{12} \\ 0 & 1 & \frac{b}{3} \\ 0 & 0 & 1 \end{bmatrix}
\]

4B. Find the determinants of each of the following matrices and write your answers in the boxes.

3M

**NOTE** $\det(3M) = 4 \cdot 3 \cdot 1 = 12$ b/c it's upper triangular.

\[
\begin{bmatrix} 4 & a & 0 \\ 4 & 3+a & b \\ 8 & 8a & 1 \end{bmatrix}
\]

N/A

this has a typo; problem discarded

\[
2M + 3I_3
\]

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WRONG: $\det(2M) + \det(3I_3)$

but $\det(A+B) = \det(A) + \det(B)$

\[
2M + 3I_3 = \begin{bmatrix} 8 & 2a & 0 \\ 0 & 6 & 2b \\ 0 & 0 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}
\]

This matrix results from $A$ by doing $R_3 \leftarrow R_3 + \frac{b}{3}R_2$ then $R_3 \leftarrow R_3 + \frac{b}{3}R_2$. None of these change the det from $\det(A)$

\[
M^{-1}M^T
\]

1

\[
\det(m^{-1} \cdot m^T) = \det(m^{-1}) \cdot \det(m^T) = \frac{1}{12} \cdot 12 = 1
\]

\[
\begin{bmatrix} 4 & a & -1 \\ 0 & 3 & b \\ 8 & 2a + 3 & b - 2 \end{bmatrix}
\]

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5. Consider the vector space $\mathbf{F}$ of functions $f : \mathbb{R} \to \mathbb{R}$ which are continuous everywhere, as discussed in class. Let $H$ consist of all members of $\mathbf{F}$ which are differentiable at $x = 1$ and furthermore, the line tangent to the graph of any member of $H$ is horizontal at $x = 1$.

5A. What does this last condition say that $h'(1)$ equals, for any member $u = h(x)$ which belongs to $H$?

\[
\boxed{h'(1) = 0}
\]

5B. Which, if any, of the following functions belong to $H$? Explain!

a) $e^{x^{-1}}$

\[
\frac{d}{dx} (e^{x^{-1}}) = e^{x^{-1}} \cdot \frac{1}{x^2} \quad \text{evaluate at } x = 1 \quad \text{to get } e^{1^{-1}} = e^0 = 1 \neq 0,
\]

so $e^{x^{-1}} \notin H$.

b) $p(x) = 2x^2 - 4x + 8$

\[
\frac{d}{dx} (2x^2 - 4x + 8) = 4x - 4 \quad \text{so } p'(1) = 4 \cdot 1 - 4 = 4 - 4 = 0; \quad \text{thus } p(x) \in H.
\]

5C. PROVE that $H$ is a subspace of $\mathbf{F}$.

\[\boxed{\hat{0} \in H?}\]

Here $\hat{0}$ is the function $f(x) = 0$; since $f'(x) = 0$ for all $x$, $f'(1) = 0$ & so $f \in H$, i.e. $\hat{0} \in H$.

(alternatively: the graph of $\hat{0}$ is \[\longrightarrow \] which has a slope of $0$ at $1$)

\[\boxed{\hat{u} + \hat{v} \in H?}\]

Since $\hat{u}, \hat{v} \in H$, $\hat{u}$ is a function satisfying $f'(1) = 0$.

\[\text{and } \hat{v} \in H, \hat{v} = \begin{array}{l} g \quad g'(1) = 0. \end{array} \]

Now $\left( f(x) + g(x) \right)' = f'(x) + g'(x)$ so when $\left( f(x) + g(x) \right)'$ is evaluated at $x = 1$ we get $f'(1) + g'(1) = 0 + 0 = 0$, so $\hat{u} + \hat{v} \in H$.

\((\text{not: it's incorrect to write } " (f(1) + g(1))' = 0." \quad \text{but } (f(x) + g(x))' \bigg|_{x = 1} \; \text{is fine})\)

\[\boxed{\alpha \hat{u} \in \mathbb{R} \; \& \; \hat{u} \in H?}\]

Since $\hat{u} \in H$, $\hat{u}$ is a function satisfying $f'(1) = 0$.

\[
\alpha \hat{u} \quad \text{is the function } \alpha f, \quad \text{and } (\alpha f)' \bigg|_{x = 1} = \alpha f'(1) = \alpha \cdot 0 = 0
\]

so $\alpha \hat{u}$ is also in $H$.
6. Let \( P_3 \) be the vector space of polynomials of degree three-or-less, and let \( H \) be all members of \( P_3 \) which have a slope of 1 when \( x = 1 \).

6A. Find three members of \( H \), one of degree one, one of degree two, and one of degree three. CIRCLE your answers

THERE ARE MANY ANSWERS; these 3 are

a) \( x \)  

Here's the technical part to say, 6A(c):
we need a polynomial of the form
\[ p(x) = ax^3 + bx^2 + cx + d \quad \text{where} \quad a \neq 0 \quad \text{and} \quad p'(1) = 1. \]

Now, \( p'(x) = 3ax^2 + 2bx + c \)  
so \( p'(1) = 3a + 2b + c \);  
so \( 3a + 2b + c = 1 \)
\[ 3a = 1 - 2b - c \]
\[ a = \frac{1}{3} - \frac{2}{3}b - \frac{1}{3} \quad \text{where} \quad b \text{ and } c \text{ are free} \quad \text{(as long as } a \neq 0) \]

for example: \( b = c = 0 \Rightarrow a = 1 \) ...

b) \( \frac{1}{2} x^2 \)

c) \( \frac{1}{3} x^3 \)

6B. Show that \( H \) is not closed under vector addition.

Let \( \tilde{u} \) be \( x \) and \( \tilde{v} \) be \( \frac{1}{2} x^2 \),  
then \( \tilde{u} \) and \( \tilde{v} \) are both in \( H \).  
Now, \( \tilde{u} + \tilde{v} \) is \( (x + \frac{1}{2} x^2) \), and the derivative of this is \( 1 + x \), which  
is \( 2 \), \ NOT \( 1 \), when \( x = 1 \), \( \therefore \tilde{u} + \tilde{v} \notin H \), and so

\( H \) is not closed under vector addition.