Math 106: Review for Exam II - SOLUTIONS

INTEGRATION TIPS

- Substitution: usually let u = a function that's "inside" another function, especially if du (possibly off by a multiplying constant) is also present in the integrand.

- Parts: $\displaystyle\int u\,dv = uv - \int v\,du$ or $\displaystyle\int uv'\,dx = uv - \int u'v\,dx$

  How to choose which part is u? Let u be the part that is higher up in the **LIATE** mnemonic below. (The mnemonics **ILATE** and **LIPET** will work equally well if you have learned one of those instead; in the latter **A** is replaced by **P**, which stands for "polynomial".)

  Logarithms (such as $\ln x$)

  Inverse trig (such as $\arctan x, \arcsin x$)

  Algebraic (such as $x, x^2, x^3 + 4$)

  Trig (such as $\sin x, \cos 2x$)

  Exponentials (such as e^x, e^{3x})

- Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

  Partial Fractions: here's an illustrative example of the setup.

$$\frac{3x^2+11}{(x+1)(x-3)^2(x^2+5)} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{Dx+E}{x^2+5}$$

  Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor $(x-3)$ on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term ($Dx + E$ here) above it on the right.

- Trigonometric Substitutions: some suggested substitutions and useful formulae follow.

Radical Form	$\sqrt{a^2-x^2}$	$\sqrt{a^2+x^2}$	$\sqrt{x^2-a^2}$
Substitution	$x = a\sin t$	$x = a\tan t$	$x = a\sec t$

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x$$

$$\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2} \qquad \cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2}$$

$$\sin(2x) = 2\sin x\cos x$$

- Powers of Trigonometric Functions: here are some strategies for dealing with these.

$\int \sin^m x \cos^n x\ dx$	Possible Strategy	Identity to Use
m odd	Break off one factor of $\sin x$ and substitute $u = \cos x$.	$\sin^2 x = 1 - \cos^2 x$
n odd	Break off one factor of $\cos x$ and substitute $u = \sin x$.	$\cos^2 x = 1 - \sin^2 x$
m, n even	Use $\sin^2 x + \cos^2 x = 1$ to reduce to only powers of $\sin x$ or only powers of $\cos x$, then use integration by parts or identities shown to right of this box.	$\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2}$ $\cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2}$

$\int \tan^m x \sec^n x \, dx$	Possible Strategy	Identity to Use
m odd	Break off one factor of $\sec x \tan x$ and substitute $u = \sec x$.	$\tan^2 x = \sec^2 x - 1$
n even	Break off one factor of $\sec^2 x$ and substitute $u = \tan x$.	$\sec^2 x = \tan^2 x + 1$
m even, n odd	Use identity at right to reduce to powers of $\sec x$ alone. Then use integration by parts or reduction formula (if allowed).	$\tan^2 x = \sec^2 x - 1$

Useful Trigonometric Derivatives and Antiderivatives

$\frac{d}{dx}\tan x = \sec^2 x$ $\qquad$ $\frac{d}{dx}\sec x = \sec x \tan x$ $\qquad$ $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

- Improper integrals: look for ∞ as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

$\lim_{x\to\infty} e^x = \infty$

$\lim_{x\to\infty} e^{-x} = 0$ $\qquad$ Note: this is the same as $\lim_{x\to-\infty} e^x$.

$\lim_{x\to\infty} 1/x = 0$ $\qquad$ Note: the answer is the same for $\lim_{x\to\infty} 1/x^2$ and similar functions.

$\lim_{x\to 0^+} 1/x = \infty$ $\qquad$ Note: the answer is the same for $\lim_{x\to 0^+} 1/x^2$ and similar functions.

$\lim_{x\to\infty} \ln x = \infty$

$\lim_{x\to 0^+} \ln x = -\infty$

$\lim_{x\to\infty} \arctan x = \pi/2$

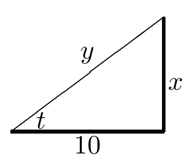
1. **Evaluate the following.**

(a) Let $u = \sin x$, so $du = \cos x \, dx$.

$$\begin{aligned}\int \sin^6 x \cos^3 x \, dx &= \int \sin^6 x(1-\sin^2 x)\cos x \, dx \\ &= \int u^6(1-u^2)\, du \\ &= \int (u^6 - u^8)\, du \\ &= \frac{u^7}{7} - \frac{u^9}{9} + C \\ &= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C\end{aligned}$$

Use $\cos^2 x = 1 - \sin^2 x$.

(b) Let $x = 10\tan t$, so $dx = 10\sec^2 t \, dt$.



$$x^2 + 10^2 = y^2 \Rightarrow y = \sqrt{x^2+100}$$

$$\sec t = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{x^2+100}}{10}$$

$$\tan t = \frac{\text{opp}}{\text{adj}} = \frac{x}{10}$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{100+x^2}} &= \int \frac{10\sec^2 t\ dt}{\sqrt{100+100\tan^2 t}} \\
&= \int \frac{10\sec^2 t\ dt}{10\sqrt{1+\tan^2 t}} && \text{Now use } 1+\tan^2 t = \sec^2 t. \\
&= \int \frac{\sec^2 t\ dt}{\sqrt{\sec^2 t}} \\
&= \int \sec t\ dt \\
&= \ln|\sec t + \tan t| + C && \text{Now use triangle above.} \\
&= \ln\left|\frac{\sqrt{x^2+100}}{10} + \frac{x}{10}\right| + C
\end{aligned}$$

(c) This is an improper integral, so we need to use a limit.

$$\begin{aligned}
\int_3^\infty \frac{1}{x(\ln x)^{100}}\,dx &= \lim_{t\to\infty}\int_3^t \frac{1}{x(\ln x)^{100}}\,dx \\
&= \lim_{t\to\infty}\int_{x=3}^{x=t} \frac{1}{u^{100}}\,du && \text{Substitute } u=\ln x, \text{so } du = \frac{dx}{x}. \\
&= \lim_{t\to\infty} \left.\frac{u^{-99}}{-99}\right|_{x=3}^{x=t} \\
&= \lim_{t\to\infty} \left.\frac{-1}{99(\ln x)^{99}}\right|_3^t \\
&= \lim_{t\to\infty}\left[\frac{-1}{99(\ln t)^{99}} - \frac{-1}{99(\ln 3)^{99}}\right] \\
&= 0 - \frac{-1}{99(\ln 3)^{99}} \\
&= \frac{1}{99(\ln 3)^{99}} && \text{So, the integral converges (to this value).}
\end{aligned}$$

(d) We'll use integration by parts: $u = x \Rightarrow du = dx$ and $dv = e^{-2x} \Rightarrow v = \frac{e^{-2x}}{-2}$.

$$\begin{aligned}
\int_0^\infty xe^{-2x}\,dx &= \lim_{t\to\infty}\int_0^t xe^{-2x}\,dx \\
&= \lim_{t\to\infty}\left[\left. x\frac{e^{-2x}}{-2}\right|_0^t - \int_0^t \frac{e^{-2x}}{-2}\,dx\right] \\
&= \lim_{t\to\infty}\left[x\frac{e^{-2x}}{-2} - \frac{e^{-2x}}{4}\right]_0^t \\
&= \lim_{t\to\infty}\left[\frac{-x}{2e^{2x}} - \frac{1}{4e^{2x}}\right]_0^t \\
&= \lim_{t\to\infty}\left[\frac{-t}{2e^{2t}} - \frac{1}{4e^{2t}}\right] - \left[\frac{0}{2e^0} - \frac{1}{4e^0}\right] \\
&= (0-0) - (0-1/4) \\
&= 1/4 && \text{So, the integral converges (to this value).}
\end{aligned}$$

(e) Partial Fractions:

Write $\frac{3x^2+2x-13}{(x-3)(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-3}$. Now multiply both sides by $(x-3)(x^2+1)$ to get

$$3x^2 + 2x - 13 = (Ax + B)(x - 3) + C(x^2 + 1).$$

Let $x = 3$. Then $20 = C(10)$, so $C = 2$.
Let $x = 0$. Then $-13 = B(-3) + 2(1)$, so $B = 5$.
Let $x = 1$. Then $-8 = (A(1) + 5)(-2) + 2(2)$, so $A = 1$.

$$\begin{aligned}
\int \frac{3x^2 + 2x - 13}{(x-3)(x^2+1)}\,dx &= \int \left[\frac{x+5}{x^2+1} + \frac{2}{x-3}\right] dx \\
&= \int \left[\frac{x}{x^2+1} + \frac{5}{x^2+1} + \frac{2}{x-3}\right] dx \qquad \text{Let } u = x^2+1, \text{ so } du = 2x dx. \\
&= \int \frac{\frac{1}{2}du}{u} + \int \left[\frac{5}{x^2+1} + \frac{2}{x-3}\right] dx \\
&= \frac{\ln u}{2} + 5\arctan x + 2\ln|x-3| + D \\
&= \frac{\ln(x^2+1)}{2} + 5\arctan x + 2\ln|x-3| + D
\end{aligned}$$

(f) Since the degree of the numerator is greater than or equal to the degree of the denominator, we do long division.

$$\begin{array}{r l}
 & 4x^2 \quad - 3x + \quad 2 + \dfrac{-5}{x-6} \\
x - 6 & \overline{\big| 4x^3 - 27x^2 + 20x - 17} \\
 & \underline{4x^3 - 24x^2} \\
 & \quad -3x^2 \\
 & \quad \underline{-3x^2 + 18x} \\
 & \qquad\qquad 2x \\
 & \qquad\qquad \underline{2x - 12} \\
 & \qquad\qquad\qquad -5
\end{array}$$

Now, we compute the integral.

$$\int \frac{4x^3 - 27x^2 + 20x - 17}{x-6}\,dx = \int \left[4x^2 - 3x + 2 - \frac{5}{x-6}\right] dx = \frac{4x^3}{3} - \frac{3x^2}{2} + 2x - 5\ln|x-6| + C$$

(g) This integral is improper at $x = 1$ because the integrand has a vertical asymptote there.

$$\begin{aligned}
\int_1^3 \frac{1}{x-1}\,dx &= \lim_{t\to 1^+} \int_t^3 \frac{1}{x-1}\,dx \\
&= \lim_{t\to 1^+} \ln|x-1|\Big|_t^3 \\
&= \lim_{t\to 1^+} [\ln|3-1| - \ln|t-1|]
\end{aligned}$$

Since $\lim\limits_{t\to 1^+} (-\ln|t-1|) = \infty$, this integral diverges (to ∞).

2. **Solve the differential equation $dy/dx = 2xy + 6x$ if the solution passes through $(0, 5)$.**

$$\frac{dy}{dx} = 2xy + 6x$$
$$\frac{dy}{dx} = 2x(y+3)$$
$$\frac{dy}{y+3} = 2x\ dx \qquad \text{Separate the variables.}$$
$$\int \frac{dy}{y+3} = \int 2x\ dx$$
$$\ln|y+3| = x^2 + C$$
$$|y+3| = e^{x^2+C} \qquad \text{Exponentiate each side to remove the ln.}$$
$$y + 3 = \pm e^C e^{x^2} \qquad |w| = z \text{ means } w = \pm z.$$
$$y = -3 + Ae^{x^2} \qquad \text{Replace } \pm e^C \text{ with } A.$$

Now we use the initial condition $y(0) = 5$ to find the value of A.

We have $5 = -3 + Ae^0 \Rightarrow A = 8$, so the solution is $y = -3 + 8e^{x^2}$.

3. **Find the second-degree Taylor polynomial for $f(x) = \sqrt{x}$ centered at $x = 100$.**

$$f(x) = x^{1/2} \qquad f(100) = 10$$
$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \qquad f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$$
$$f''(x) = \frac{-1}{4}x^{-3/2} = \frac{-1}{4x^{3/2}} \qquad f''(100) = \frac{-1}{4 \cdot 100^{3/2}} = \frac{-1}{4000}$$

$$P_2(x) = f(100) + f'(100)(x-100) + \frac{f''(100)}{2!}(x-100)^2$$
$$= 10 + \frac{x-100}{20} - \frac{(x-100)^2}{8000}$$

4. **What is the maximum possible error that can occur in your Taylor approximation from the previous problem on the interval [100, 110]?**

We know that $|f(x) - P_n(x)| \le \dfrac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}$.

In this case, $n = 2$, $x_0 = 100$, and $x = 110$ (the farthest from x_0 that we are considering).

$K_3 = \text{max of } |f'''(x)| \text{ on } [100, 110] = \text{max of } |\dfrac{3}{8x^{5/2}}| \text{ on } [100, 110] = \dfrac{3}{8 \cdot 100^{5/2}} = \dfrac{3}{800,000}$

Putting this all together, we have $|f(x) - P_2(x)| \le \dfrac{\frac{3}{800,000}}{3!}|110 - 100|^3 = \dfrac{1}{1600}$.

5. **Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.**

(a) $\displaystyle\int_1^\infty \frac{6 + \cos x}{x^{0.99}}\, dx$

For all $x \ge 1$, we have $\dfrac{6 + \cos x}{x^{0.99}} \ge \dfrac{6-1}{x^{0.99}} = \dfrac{5}{x^{0.99}}$ because the minimum value of $\cos x$ is -1.

Since $\displaystyle\int_1^\infty \frac{5\ dx}{x^{0.99}}$ diverges (compute yourself or notice that $p = 0.99 < 1$), we know that the integral in question must diverge too.

(b) $\int_1^\infty \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1}\, dx$

For all $x \geq 1$, we have $\frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \leq \frac{4x^3}{x^5} = 4\frac{1}{x^2}$. (We've made the denominator smaller and the numerator larger, so the new fraction is larger.)

$$\begin{aligned}
4\int_1^\infty \frac{dx}{x^2} &= 4\lim_{t\to\infty}\int_1^t \frac{dx}{x^2} \\
&= 4\lim_{t\to\infty} \left.\frac{-1}{x}\right|_1^t \\
&= 4\lim_{t\to\infty} \left[\frac{-1}{t} - \frac{-1}{1}\right] \\
&= 4[0-(-1)] \\
&= 4
\end{aligned}$$

Therefore, the original integral in question must converge to a value less than 4.