

## Math 106: Review for Exam II

### INTEGRATION TIPS

- Substitution: usually let  $u =$  a function that's "inside" another function, especially if  $du$  (possibly off by a multiplying constant) is also present in the integrand.

- Parts:  $\int u dv = uv - \int v du$  or  $\int uv' dx = uv - \int u'v dx$

How to choose which part is  $u$ ? Let  $u$  be the part that is higher up in the **LIATE** mnemonic below. (The mnemonics **ILATE** and **LIPET** will work equally well if you have learned one of those instead; in the latter **A** is replaced by **P**, which stands for "polynomial".)

**L**ogarithms (such as  $\ln x$ )

**I**nverse trig (such as  $\arctan x, \arcsin x$ )

**A**lgebraic (such as  $x, x^2, x^3 + 4$ )

**T**rig (such as  $\sin x, \cos 2x$ )

**E**xponentials (such as  $e^x, e^{3x}$ )

- Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

Partial Fractions: here's an illustrative example of the setup.

$$\frac{3x^2 + 11}{(x + 1)(x - 3)^2(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 5}$$

Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor  $(x - 3)$  on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term ( $Dx + E$  here) above it on the right.

- Trigonometric Substitutions: some suggested substitutions and useful formulae follow.

Radical Form	$\sqrt{a^2 - x^2}$	$\sqrt{a^2 + x^2}$	$\sqrt{x^2 - a^2}$
Substitution	$x = a \sin t$	$x = a \tan t$	$x = a \sec t$

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 & \tan^2 x + 1 &= \sec^2 x \\ \sin^2 x &= \frac{1 - \cos(2x)}{2} & \cos^2 x &= \frac{1 + \cos(2x)}{2} \\ \sin(2x) &= 2 \sin x \cos x \end{aligned}$$

- Powers of Trigonometric Functions: here are some strategies for dealing with these.

$\int \sin^m x \cos^n x dx$	Possible Strategy	Identity to Use
$m$ odd	Break off one factor of $\sin x$ and substitute $u = \cos x$ .	$\sin^2 x = 1 - \cos^2 x$
$n$ odd	Break off one factor of $\cos x$ and substitute $u = \sin x$ .	$\cos^2 x = 1 - \sin^2 x$
$m$ even AND $n$ even	Use $\sin^2 x + \cos^2 x = 1$ to reduce to only powers of $\sin x$ or only powers of $\cos x$ , then use integration by parts or identities shown to right of this box.	$\sin^2 x = \frac{1 - \cos(2x)}{2}$ $\cos^2 x = \frac{1 + \cos(2x)}{2}$

$\int \tan^m x \sec^n x dx$	Possible Strategy	Identity to Use
$m$ odd	Break off one factor of $\sec x \tan x$ and substitute $u = \sec x$ .	$\tan^2 x = \sec^2 x - 1$
$n$ even	Break off one factor of $\sec^2 x$ and substitute $u = \tan x$ .	$\sec^2 x = \tan^2 x + 1$
$m$ even AND $n$ odd	Use identity at right to reduce to powers of $\sec x$ alone. Then use integration by parts or reduction formula (if allowed).	$\tan^2 x = \sec^2 x - 1$

Useful Trigonometric Derivatives and Antiderivatives

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\int \sec x dx = \ln |\sec x + \tan x| + C$$

- Improper integrals: look for  $\infty$  as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

$$\lim_{x \rightarrow \infty} e^x =$$

$$\lim_{x \rightarrow \infty} e^{-x} =$$

$$\lim_{x \rightarrow \infty} 1/x =$$

$$\lim_{x \rightarrow 0^+} 1/x =$$

$$\lim_{x \rightarrow \infty} \ln x =$$

$$\lim_{x \rightarrow 0^+} \ln x =$$

$$\lim_{x \rightarrow \infty} \arctan x =$$

Note: this is the same as  $\lim_{x \rightarrow -\infty} e^x$

Note: the answer is the same for  $\lim_{x \rightarrow \infty} 1/x^2$  and similar functions

Note: the answer is the same for  $\lim_{x \rightarrow 0^+} 1/x^2$  and similar functions

- Evaluate the following.

(a)  $\int \sin^6 x \cos^3 x dx$

(b)  $\int \frac{dx}{\sqrt{100 + x^2}}$

$$(c) \int_3^{\infty} \frac{1}{x(\ln x)^{100}} dx$$

$$(d) \int_0^{\infty} xe^{-2x} dx$$

$$(e) \int \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} dx$$

$$(f) \int \frac{4x^3 - 27x^2 + 20x - 17}{x - 6} dx$$

$$(g) \int_1^3 \frac{1}{x - 1} dx$$

2. Solve the differential equation  $dy/dx = 2xy + 6x$  if the solution passes through  $(0, 5)$ .

3. Find the second-degree Taylor polynomial for  $f(x) = \sqrt{x}$  centered at  $x = 100$ .

4. What is the maximum possible error that can occur in your Taylor approximation from the previous problem on the interval  $[100, 110]$ ?

5. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.

(a) 
$$\int_1^{\infty} \frac{6 + \cos x}{x^{0.99}} dx$$

(b) 
$$\int_1^{\infty} \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} dx$$