1. Let $L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$; then $LU = A = \begin{bmatrix} 1 & 5 & 3 & 0 \\ 4 & 22 & 12 & 4 \\ 0 & -6 & 1 & -12 \end{bmatrix}$.

Following the method developed in class, use this LU factorization to solve $A\mathbf{x} = \begin{bmatrix} 20 \\ 108 \\ -84 \end{bmatrix}$.

2. Suppose A is a 4-by-5 matrix with column vectors $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, $\mathbf{c}_4$, $\mathbf{c}_5$, and the solutions to $A\mathbf{x} = \mathbf{0}$ are given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 8 \\ 0 \\ 6 \\ 0 \\ 1 \end{bmatrix}, \text{ where } x_2 \text{ and } x_5 \text{ are free.}$$

2a. Can $\mathbf{c}_1$ be written as a linear combination of the other columns? Show an explicit combination or explain why there are none.

2b. Can $\mathbf{c}_4$ be written as a linear combination of the other columns? Show an explicit combination or explain why there are none.

2c. Are the four columns $\mathbf{c}_2$, $\mathbf{c}_3$, $\mathbf{c}_4$, and $\mathbf{c}_5$, linearly independent? Explain your answer.

2d. Do the columns of A span $\mathbf{R}^4$? Explain. (Hint: you should be able to figure out how many pivot elements there are. Are there any rows of 0's in the RREF of A?)

3. Let $D = \begin{bmatrix} 11 & 14 & 7 & 2 \\ 3 & 29 & 8 & 0 \\ 0 & 3 & 0 & 0 \\ 5 & 17 & 12 & 0 \end{bmatrix}$.

3a. Find $\det(D)$. Use cofactor expansion across whatever rows or columns you want, but choose smartly.

3b. Does the inverse of D exist? If so, what's $\det(D^{-1})$? If not, explain why D^{-1} doesn't exist.

3c. Is D singular? Explain what this means.

4. Find the inverse of the matrix $B = \begin{bmatrix} 1 & 4 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ using the method we developed in class: Apply to I_3 the same elementary row operations we apply to B ....

5. Consider the Linear Transformations T from $\mathbf{R}^m$ to $\mathbf{R}^n$ defined by $T(\mathbf{u}) = A\mathbf{u}$ where

$$A = \begin{bmatrix} 2 & 0 \\ 5 & 1 \end{bmatrix}.$$

and S from $\mathbf{R}^p$ to $\mathbf{R}^q$ defined by $S(\mathbf{v}) = Z\mathbf{v}$ where

$$Z = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 4 & 0 \end{bmatrix}.$$

5a. What are the values of m, n, p and q?

5b. The composition $T \circ S$ is a linear transformation from $\mathbf{R}^i$ to $\mathbf{R}^j$, where the values of i and j are what?

5c. Find $(T \circ S)\left(\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}\right)$.

5d. Compute the matrix P which gives the values of $T \circ S$ directly, ie, so that $(T \circ S)\mathbf{v} = P\mathbf{v}$.

6a. What are the two "rules" that a linear transformation T from $\mathbf{R}^m$ to $\mathbf{R}^n$ must satisfy?

6b. If $T\left(\begin{bmatrix} 10 \\ 20 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, find $T\left(\begin{bmatrix} 34 \\ 64 \end{bmatrix}\right)$.

HINT: Write $\begin{bmatrix} 34 \\ 64 \end{bmatrix}$ as $\alpha \begin{bmatrix} 10 \\ 20 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; the values of α and β are obvious. Compute $T\left(\begin{bmatrix} 34 \\ 64 \end{bmatrix}\right)$ using your result.