Answer Key for Exam #1

1. The vectors $\begin{pmatrix}3\\0\\4\\1\\1\\3\end{pmatrix}$ and $\begin{pmatrix}1\\1\\1\\4\\4\\1\end{pmatrix}$ each have have length 6:

$$\sqrt{3^2+0^2+4^2+1^2++1^2+3^2} = 6 = \sqrt{1^2+1^2+1^2+4^2+4^2+1^2}.$$

Their dot product is $3+0+4+4+4+3=18$, so the angle θ between them satisfies $\cos\theta = \dfrac{18}{6\cdot 6} = \dfrac{1}{2}$. Therefore $\theta = \frac{\pi}{3}$, or $60°$.

2. We use elimination on an augmented matrix. First two steps: subtract twice the first row from the second, and three times the first row from the third. Next two steps: multiply the second row by -1, and then add five times the second row to the third. This gives

$$\begin{pmatrix}1&2&3&&10\\2&3&1&&7\\3&1&2&&19\end{pmatrix} \longrightarrow \begin{pmatrix}1&2&3&&10\\0&-1&-5&&-13\\0&-5&-7&&-11\end{pmatrix} \longrightarrow \begin{pmatrix}1&2&3&&10\\0&1&5&&13\\0&0&18&&54\end{pmatrix}.$$

The third equation gives $x_3 = 3$. Then the second equation becomes $x_2 + 15 = 13$, so $x_2 = -2$. Finally the first equation becomes $x_1 - 4 + 9 = 10$, so $x_1 = 5$.

3. The inverse of $\left(\begin{smallmatrix}2&3\\3&4\end{smallmatrix}\right)$ is

$$\frac{1}{2\cdot 4 - 3\cdot 3}\begin{pmatrix}4&-3\\-3&2\end{pmatrix} = \begin{pmatrix}-4&3\\3&-2\end{pmatrix},$$

and the inverse of $\left(\begin{smallmatrix}3&4\\4&5\end{smallmatrix}\right)$ is

$$\frac{1}{3\cdot 5 - 4\cdot 4}\begin{pmatrix}5&-4\\-4&3\end{pmatrix} = \begin{pmatrix}-5&4\\4&-3\end{pmatrix},$$

so you might guess that $A^{-1} = \begin{pmatrix}-4&0&0&3\\0&-5&4&0\\0&4&-3&0\\3&0&0&-2\end{pmatrix}$, and once guessed this is easy to check. Alternatively we can use row operations to reduce $[A \;\; I]$ to $\left[I \;\; A^{-1}\right]$. I'll try to avoid fractions: subtract the first row from the fourth, and the second row from the third to get

$$\begin{pmatrix}2&0&0&3&&1&0&0&0\\0&3&4&0&&0&1&0&0\\0&4&5&0&&0&0&1&0\\3&0&0&4&&0&0&0&1\end{pmatrix} \longrightarrow \begin{pmatrix}2&0&0&3&&1&0&0&0\\0&3&4&0&&0&1&0&0\\0&1&1&0&&0&-1&1&0\\1&0&0&1&&-1&0&0&1\end{pmatrix}.$$

Next, subtract twice the fourth row from the first, and three times the third row from the second:

$$\begin{pmatrix}2&0&0&3&&1&0&0&0\\0&3&4&0&&0&1&0&0\\0&1&1&0&&0&-1&1&0\\1&0&0&1&&-1&0&0&1\end{pmatrix} \longrightarrow \begin{pmatrix}0&0&0&1&&3&0&0&-2\\0&0&1&0&&0&4&-3&0\\0&1&1&0&&0&-1&1&0\\1&0&0&1&&-1&0&0&1\end{pmatrix}.$$

Now subtract the second row from the third and the first row from the fourth:

$$\begin{pmatrix}0&0&0&1&&3&0&0&-2\\0&0&1&0&&0&4&-3&0\\0&1&1&0&&0&-1&1&0\\1&0&0&1&&-1&0&0&1\end{pmatrix} \longrightarrow \begin{pmatrix}0&0&0&1&&3&0&0&-2\\0&0&1&0&&0&4&-3&0\\0&1&0&0&&0&-5&4&0\\1&0&0&0&&-4&0&0&3\end{pmatrix}.$$

If we now switch the first and fourth rows, and also the second and third rows, we get A^{-1} as before.

4(a) Step 1: subtract the first row of A from all the others. Step 2: subtract twice the second row from the third, and three times the second row from the fourth. Step 3: subtract three times the third row from the fourth. This gives

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = U.$$

Step 1 puts 1's in the second through fourth rows of column 1 of L; step 2 puts 2 and 3 respectively in the third and fourth rows of column 2; and step 3 puts a 3 in the fourth row and third column of L, so $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}$, $U = L^T$, and $A = LU = LL^T$.

4(b) We may now solve $A\vec{x} = \begin{pmatrix} 1 \\ 8 \\ 27 \\ 64 \end{pmatrix}$ in two steps. First we solve $L\vec{y} = \vec{b}$, which in this case is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 27 \\ 64 \end{pmatrix},$$

and we get $y_1 = 1$; $1 + y_2 = 8$, so $y_2 = 7$; $1 + 14 + y_3 = 27$, so $y_3 = 12$; and $1 + 21 + 36 + y_4 = 64$ so $y_4 = 6$. Finally we solve $U\vec{x} = \vec{y}$, which in this case is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 12 \\ 6 \end{pmatrix}.$$

Here we have $x_4 = 6$; $x_3 + 18 = 12$ so $x_3 = -6$; $x_2 - 12 + 18 = 7$ so $x_2 = 1$; $x_1 + 1 - 6 + 6 = 1$ so $x_1 = 0$.

4(c) Here we have to reduce $[L \;\; I]$ to $\left[I \;\; L^{-1}\right]$ by row operations, which is not too hard:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 3 & 1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 2 & -3 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{pmatrix}$$

So $\quad L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$, and $\quad U^{-1} = \left(L^T\right)^{-1} = \left(L^{-1}\right)^T \quad$ so $\quad U^{-1} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

4(d) Since $A = LU$ and L and U are both invertible, we have $A^{-1} = U^{-1}L^{-1}$. Therefore

$$A^{-1} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -6 & 4 & -1 \\ -6 & 14 & -11 & 3 \\ 4 & -11 & 10 & -3 \\ -1 & 3 & -3 & 1 \end{pmatrix}.$$

5. If $R = \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}$ and $P = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \end{pmatrix}$, then

$$\begin{aligned} RP &= \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos 2\phi \, \cos\phi + \sin 2\phi \, \sin\phi \\ \sin 2\phi \, \cos\phi - \cos 2\phi \, \sin\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos(2\phi - \phi) \\ \sin(2\phi - \phi) \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \end{pmatrix} = P, \end{aligned}$$

and also

$$\begin{aligned} PR &= \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \end{pmatrix} \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \begin{pmatrix} \cos 2\phi \, \cos\phi + \sin 2\phi \, \sin\phi & \sin 2\phi \, \cos\phi - \cos 2\phi \, \sin\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \begin{pmatrix} \cos(2\phi - \phi) & \sin(2\phi - \phi) \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \end{pmatrix} = P. \end{aligned}$$

If P projects a vector onto a line and R reflects it through the same line, then we should expect $RP = P$, because R is acting on a vector that has already been projected onto the line by P, and R does nothing to such a vector. (Note: this is the explanation for RP, not for PR, because the matrix on the right sees the vector first.) We also expect $PR = P$, because the projection of a vector onto a line is the same as the projection of its reflection in that line.

Another, less trigonometrically intensive way to do the problem is to recall that P was defined to be $\frac{1}{2}(I + R)$. If we solve this for R we get $R = 2P - I$. Therefore

$$RP = (2P - I)\,P = 2P^2 - P = 2P - P = P \quad \text{and} \quad PR = P\,(2P - I) = 2P^2 - P = 2P - P = P$$

since a projection matrix is its own square.

6. If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $AA^T = A^TA$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Multiplying this out we get

$$\begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix},$$

which implies that $a^2 + b^2 = a^2 + c^2$, $ac + bd = ab + cd$ and $c^2 + d^2 = b^2 + d^2$. The first and third of these equations tell us that $b^2 = c^2$, so $b = \pm c$. We can rewrite the second equation as

$$\begin{aligned} 0 &= ac + bd - ab - cd \\ &= a(c - b) + d(b - c) \\ &= (a - d)(c - b), \end{aligned}$$

so either $b = c$ or $a = d$. There are then two possibilities to satisfy all the equations: either $b = c$; or $a = d$ and $b = -c$. If $b = c$ then A is symmetric, in which case it is obvious that $AA^T = A^T A$ since both sides equal A^2. If $a = d$ and $b = -c$ then A looks like

$$\begin{pmatrix} a & -c \\ c & a \end{pmatrix}$$

This would be a rotation matrix if $a^2 + c^2 = 1$, but $a^2 + c^2$ won't be 1 in general. However, it will equal some nonnegative number, which we might as well call r^2 for some nonnegative r. Then we have

$$\left(\frac{a}{r}\right)^2 + \left(\frac{c}{r}\right)^2 = 1,$$

so $(\frac{a}{r}, \frac{c}{r})$ is a point on the unit circle, and therefore there is an angle θ such that $\frac{a}{r} = \cos\theta$ and $\frac{c}{r} = \sin\theta$. Then

$$\begin{pmatrix} a & -c \\ c & a \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where

$$r = \sqrt{a^2 + c^2}, \quad \cos\theta = \frac{a}{r}, \quad \text{and} \quad \sin\theta = \frac{c}{r}.$$

Therefore A is a multiple of a rotation matrix.

Scores: The median and mean were both 88.

Score	Frequency
98	1
95	2
94	1
93	2
92	2

Score	Frequency
91	2
89	1
88	3
87	1
86	1

Score	Frequency
85	2
83	2
78	1
76	1
75	1