

1. Suppose the solutions of a matrix equation $Ax = b$ are written in the form $p + v_h$, where p is a particular solution of $Ax = b$ and v_h gives all solutions of the corresponding homogeneous equation $Ax = 0$.

Suppose $b = \begin{bmatrix} 2 \\ -13 \\ 0 \\ 2009 \end{bmatrix}$, $p = \begin{bmatrix} 11 \\ 0 \\ -2 \\ 5 \\ 0 \end{bmatrix}$ and $v_h = x_2 \begin{bmatrix} -9 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 7 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, where x_2 and x_5 are free.

1a. How many rows does A have? (4) How many columns? (5) because the solutions have 5 entries, x_1, \dots, x_5 where x_i is the weight of the i th column of A

1b. Label the columns of A as c_1, c_2, \dots, c_p . Is the set $S = \{c_1, c_2, \dots, c_p\}$ linearly independent? Explain in terms of the definition of linear independence.

NOTE WELL! A itself is "sight unseen"; we do NOT know the actual entries of A .

But we DO know there are infinitely many solns to $Ax = 0$ since x_2 & x_5 are free,

and this means there are ways to write $x_1c_1 + x_2c_2 + x_3c_3 + x_4c_4 + x_5c_5 = 0$ besides the trivial solution $x_1 = x_2 = \dots = x_5 = 0$, so S is NOT L.I. (this is where the definition comes in !!)

1c. Write c_3 as a linear combination of the other columns, or explain why this cannot be done.

We can do this since it is possible to find a soln of

$$x_1c_1 + x_2c_2 + x_3c_3 + x_4c_4 + x_5c_5 = 0 \text{ in which } x_3 \neq 0:$$

for example, take $x_2 = 0$ and $x_5 = 1$ in v_h ; then

$$x_1 = -9x_2 + 7x_5 = 7; \quad x_3 = -2x_5 = -2; \quad x_4 = 0 \text{ and so}$$

$$7c_1 + 0c_2 - 2c_3 + 0c_4 + 1c_5 = 0; \text{ solve for } c_3 \text{ to get}$$

$$c_3 = \frac{7}{2}c_1 + \frac{1}{2}c_5 \text{ (other answers are possible through different choices of } x_2 \text{ & } x_5)$$

NOTE WELL: we DON'T know A , and we WERE NOT ASKED for this, BUT we CAN find that

$$\text{RREF}(A|b) = \left[\begin{array}{cccc|c} 1 & 9 & 0 & 0 & -7 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

1d. Write c_4 as a linear combination of the other columns, or explain why this cannot be done.

This cannot be done. For suppose otherwise. Then

$$c_4 = \alpha_1c_1 + \alpha_2c_2 + \alpha_3c_3 + \alpha_5c_5 \text{ for some}$$

scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_5$ [it doesn't matter WHAT they are]

So, $0 = \alpha_1c_1 + \alpha_2c_2 + \alpha_3c_3 - 1c_4 + \alpha_5c_5$ This represents a soln of $Ax = 0$ in which the weight of c_4 is non-zero. BUT v_h gives ALL solns of $Ax = 0$, and in v_h , x_4 is ALWAYS 0, never -1, a contradiction. So c_4 is NOT a L.C. of the other columns.

ALTERNATE but TRICKY COLN P IC → Using this you CAN argue that $[[c_1, c_2, c_4, c_5] | c_3]$ or $\left[\begin{array}{cccc|c} c_1 & c_2 & c_4 & c_5 & c_3 \end{array} \right]$

but this is not quite RREF'd:

$$\left[\begin{array}{cccc|c} 1 & 9 & 0 & 0 & -7 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

NOTE we only x_2 is free. take it as 0 put $x_1 = 7/2, x_2 = 0, x_4 = 0, x_5 = 1/2$

1e. Show how to express b as a linear combination of all p columns in such a way that none of the weights involved are 0.

This means, find a soln of $Ax = b$ in which none of the weights in x are 0.

Try $x_2 = x_5 = 1$; then $x = p + v_h = \begin{bmatrix} 11 \\ 0 \\ -2 \\ 5 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -9 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 7 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ -4 \\ 5 \\ 1 \end{bmatrix}$, that is,

$$b = 9c_1 + 1c_2 - 4c_3 + 5c_4 + 1c_5 \text{ and none of the weights are 0.}$$

2. Suppose $T: \mathbb{R}^a \rightarrow \mathbb{R}^z$ is a transformation. Give the definitions of each of the following:

2a. T is a linear transformation.

The following 2 conditions must be met:

$$\textcircled{1} T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \text{ for all } \vec{u}, \vec{v} \text{ in } \mathbb{R}^a$$

$$\textcircled{2} T(\alpha \vec{u}) = \alpha T(\vec{u}) \text{ for all } \vec{u} \in \mathbb{R}^a \text{ \& all scalars } \alpha.$$

2b. T is onto \mathbb{R}^z .

T is onto $\mathbb{R}^z \iff$ given ANY $\vec{b} \in \mathbb{R}^z$, there is at least one \vec{x} in \mathbb{R}^a for which $T(\vec{x}) = \vec{b}$.

2c. Suppose $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2x_3 + 2x_1 \\ 0 \\ x_1 + x_2 + x_3 \\ 2x_2 + 7 \end{bmatrix}$. Show by example that T is not

a linear transformation and that it actually fails both parts of the definition in (2a).

Does $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) + T\left(\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right)$? NO, as follows:

$$\text{The left side is } T\left(\begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}\right) = \begin{bmatrix} 7 \cdot 9 + 2 \cdot 5 \\ 0 \\ 5 + 7 + 9 \\ 2 \cdot 7 + 7 \end{bmatrix} = \begin{bmatrix} 63 + 10 \\ 0 \\ 21 \\ 21 \end{bmatrix} = \begin{bmatrix} 73 \\ 0 \\ 21 \\ 21 \end{bmatrix}$$

$$\text{and the right side is } T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) + T\left(\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ 0 \\ 6 \\ 11 \end{bmatrix} + \begin{bmatrix} 38 \\ 0 \\ 15 \\ 17 \end{bmatrix} = \begin{bmatrix} 46 \\ 0 \\ 21 \\ 28 \end{bmatrix}$$

$$\text{and } \begin{bmatrix} 73 \\ 0 \\ 21 \\ 21 \end{bmatrix} \neq \begin{bmatrix} 46 \\ 0 \\ 21 \\ 28 \end{bmatrix}$$

Does $T\left(10 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = 10 T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$? No as follows:

$$T\left(10 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}\right) = \begin{bmatrix} 620 \\ 0 \\ 60 \\ 47 \end{bmatrix} \text{ where as } 10 T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = 10 \begin{bmatrix} 8 \\ 0 \\ 6 \\ 11 \end{bmatrix} = \begin{bmatrix} 80 \\ 0 \\ 60 \\ 110 \end{bmatrix}$$

[Note there are co-many examples one can give here!]

3. Suppose $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ is the linear transformation whose standard matrix is $A = \begin{bmatrix} 5 & 11 & 23 \\ 5 & 14 & 17 \\ 2 & 6 & 6 \\ 3 & 8 & 11 \end{bmatrix}$.

3a. Find $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$. $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} =$ "one copy of col 1 + one copy of col. 3"

3b. Let $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ be a vector in \mathbb{R}^4 . What, if any, conditions on b_1, \dots, b_4 must be met to guarantee

\mathbf{b} is in the range of T ? Show any matrices you use in answering this question.

We need to know conditions on b_1, \dots, b_4 for which the system represented by $A\vec{x} = \vec{b}$ is consistent; a "super augmented" matrix keeps track of b_1, \dots, b_4 as $A \rightarrow \text{RREF}(A)$:

$$\left[\begin{array}{ccc|cccc} x_1 & x_2 & x_3 & b_1 & b_2 & b_3 & b_4 \\ 5 & 11 & 23 & 1 & 0 & 0 & 0 \\ 5 & 14 & 17 & 0 & 1 & 0 & 0 \\ 2 & 6 & 6 & 0 & 0 & 1 & 0 \\ 3 & 8 & 11 & 0 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|cccc} 1 & 0 & 9 & 0 & 0 & -4 & 3 \\ 0 & 1 & -2 & 0 & 0 & 3/2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 7/2 & -4 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 \end{array} \right] \text{ shows the system rep'd}$$

$$\text{by } A\vec{x} = \vec{b} \text{ is consistent} \Leftrightarrow \begin{cases} 0 = b_1 + 7/2 b_3 - 4b_4 \\ 0 = b_2 - b_3 - b_4 \end{cases}$$

3c. Suppose $\mathbf{d} = \begin{bmatrix} 3 \\ 12 \\ d_3 \\ d_4 \end{bmatrix}$. Use the conditions in (3b) to find all values of d_3 and d_4 for which \mathbf{d} is in

the range of T . (Note you will be setting up a little linear system, and you should use our linear algebra techniques to solve it).

we need $\begin{cases} 0 = 3 + 7/2 d_4 - 4d_4 \\ 0 = 12 - d_3 - d_4 \end{cases}$ The corresponding augmented matrix is:

$$\left[\begin{array}{cc|c} 7/2 & -4 & -3 \\ -1 & -1 & -12 \end{array} \right] \text{ which has RREF } \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 6 \end{array} \right]; \text{ i.e., } \vec{d} = \begin{bmatrix} 3 \\ 12 \\ 6 \\ 6 \end{bmatrix}$$

3d. Is T onto \mathbb{R}^4 ? Explain your answer.

No; unless the conditions in 3b are met, there will be no \vec{x} s.t. $T(\vec{x}) = \vec{b}$.

($A\vec{x} = \vec{b}$ will represent an inconsistent system if those conditions are not met.)

3e. Is T one-to-one? Explain your answer.

No. Since x_3 is free, there are ∞ -many solutions to $T(\vec{x}) = \vec{0}$.

(thus it is possible for $T(\vec{u}) = T(\vec{v})$ yet $\vec{u} \neq \vec{v}$)

4. Let $F = \begin{bmatrix} 2 & -3 & w \\ 1 & 5 & -2 \end{bmatrix}$ and $G = \begin{bmatrix} 8 & x & -6 \\ 7 & -2 & 1 \\ 4 & y & -5 \end{bmatrix}$, and $P = \begin{bmatrix} 7 & 23 & z \\ q & -12 & 9 \end{bmatrix}$; suppose $FG = P$.

Show all your work in the following:

4a. Find q . q is the "row 2, col 1" entry of P

so we multiply (row 2 of F) by (col. 1 of G) to find $q = 1 \cdot 8 + 5 \cdot 7 - 2 \cdot 4$
 $= 8 + 35 - 8$
 $= 35$

4b. Find w . (row 1 of F) * (col 1 of G) = 7 is the "best" to use, because ONLY the unknown w appears:

$$2 \cdot 8 - 3 \cdot 7 + 4w = 7 \Rightarrow 16 - 21 + 4w = 7 \Rightarrow 4w = 7 - 16 + 21 = 12 \Rightarrow w = 3$$

4c. Find z .

$$\begin{aligned} (\text{row 1 of } F) * (\text{col 3 of } G) &= z \Rightarrow z = (2)(-6) + (-3)(1) + (w)(-5); \text{ since } w=3, \\ z &= -12 - 3 - 15 \\ &= -30 \end{aligned}$$

4d. Find x and y . Use linear algebra techniques to solve any system this problem requires.

$$(\text{row 1 of } F) * (\text{col 2 of } G) = 23$$

$$2x + 6 + 3y = 23$$

$$\boxed{2x + 3y = 17}$$

$$(\text{row 2 of } F) * (\text{col 2 of } G) = -12$$

$$x - 10 - 2y = -12$$

$$\boxed{x - 2y = -2}$$

We have 2 eqns & 2 unknowns. Solved by $\begin{bmatrix} 2 & 3 & | & 17 \\ 1 & -2 & | & -2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & | & 4 \\ 0 & 1 & | & 3 \end{bmatrix}$$

$$\Rightarrow x=4, y=3$$

5. Suppose that A is a 2×2 matrix and the following row operations convert A into I_2 : First, rows 1 and 2 of A are swapped. Then 4 copies of row 1 are subtracted from row 2. Finally, row 2 is multiplied by 5.

5a. What three elementary matrices E_1 , E_2 , and E_3 represent these three row operations, respectively?

Remember! An elementary matrix is one that is created from I by a SINGLE row operation; an elementary matrix is "one step away" from I

5b. Use the elementary matrices to find A^{-1} .

To say these three operations "turn A into I_2 "

$$\text{means } E_3 (E_2 (E_1 A)) = I_2, \text{ so } (E_3 E_2 E_1) A = I_2 \therefore E_3 E_2 E_1 = A^{-1}$$

$$\text{And } E_3 E_2 E_1 = \begin{bmatrix} 0 & 1 \\ 5 & -20 \end{bmatrix}$$

5c. Find A .

$$A = (A^{-1})^{-1} = \left(\begin{bmatrix} 0 & 1 \\ 5 & -20 \end{bmatrix} \right)^{-1} = \frac{1}{0(-20) - 5 \cdot 1} \begin{bmatrix} -20 & -1 \\ -5 & 0 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} -20 & -1 \\ -5 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 1/5 \\ 1 & 0 \end{bmatrix}$$

5d. What is the determinant of A ?

$$\text{"ad-bc"} \left[\underline{\text{NOT}} \frac{1}{\text{ad-bc}} \right] \text{ is } 4 \cdot 0 - 1 \cdot \frac{1}{5} = -\frac{1}{5}$$

5e. Is A singular or nonsingular?

remember, our mnemonic for this was, that

to have $\det(A) = 0$ was a "singular" [rare] occurrence.

SO A is non singular.