1. Suppose the solutions of a matrix equation $A\mathbf{x} = \mathbf{b}$ are written in the form $\mathbf{p} + \mathbf{v}_h$, where $\mathbf{p}$ is a particular solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}_h$ gives all solutions of the corresponding homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Suppose $\mathbf{b} = \begin{bmatrix} 2 \\ -13 \\ 0 \\ 2009 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 11 \\ 0 \\ -2 \\ 5 \\ 0 \end{bmatrix}$ and $\mathbf{v}_h = x_2 \begin{bmatrix} -9 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 7 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, where x_2 and x_5 are free.

1a. How many *rows* does A have? 4 How many *columns*? 5 because the solutions have 5 entries, $x_1, \dots, x_5$ where x_i is the weight of the ith column of A

(since $\vec{b}$ is a L.C. of the columns of A & $\vec{b}$ has 4 entries)

1b. Label the columns of A as $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_p$. Is the set $S = \{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_p\}$ *linearly independent*? Explain in terms of the definition of linear independence.

NOTE WELL! A itself is "sight unseen"; we do <u>NOT</u> know the actually entries of A.

But we <u>do</u> know there are infinitely many soln's to $A\vec{x} = \vec{0}$ since x_2 & x_5 are free,

and <u>this</u> means there are ways to write $\underbrace{x_1\vec{c}_1 + x_2\vec{c}_2 + x_3\vec{c}_3 + x_4\vec{c}_4 + x_5\vec{c}_5 = \vec{0}}$ besides the trivial solution $x_1 = x_2 = \dots = x_5 = 0$, so

S is <u>NOT</u> L.I. (<u>this</u> is where the definition comes in !!)

1c. Write $\mathbf{c}_3$ as a linear combination of the other columns, or explain why this cannot be done.

We <u>can</u> do this since it is possible to find a soln of

$x_1\vec{c}_1 + x_2\vec{c}_2 + x_3\vec{c}_3 + x_4\vec{c}_4 + x_5\vec{c}_5 = \vec{0}$ in which $x_3 \neq 0$:

For example, take $x_2 = 0$ and $x_5 = 1$ in $\vec{v}_h$; then

$x_1 = -9x_2 + 7x_5 = 7$; $x_3 = -2x_5 = -2$; $x_4 = 0$ and so

$7\vec{c}_1 + 0\vec{c}_2 + {}^{-}2\vec{c}_3 + 0\vec{c}_4 + 1\vec{c}_5 = \vec{0}$; solve for $\vec{c}_3$ to get

$\vec{c}_3 = 7/2\,\vec{c}_1 + 1/2\,\vec{c}_5$ (other answers are possible through different choices of x_2 & x_5)

NOTE WELL: we DON'T know A, and we WERE <u>NOT</u> ASKED for this, <u>BUT</u> we <u>can</u> find that

$$\text{RREF}(A|\vec{b}) = \left[\begin{array}{ccccc|c} 1 & 9 & 0 & 0 & -7 & 11 \\ 0 & 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1d. Write $\mathbf{c}_4$ as a linear combination of the other columns, or explain why this cannot be done.

This <u>cannot</u> be done. For suppose otherwise. Then

$\vec{c}_4 = \alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 + \alpha_3\vec{c}_3 + \alpha_5\vec{c}_5$ for some

scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_5$ [it doesn't matter WHAT they are]

SO, $\vec{0} = \alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 + \alpha_3\vec{c}_3 + {-1}\vec{c}_4 + \alpha_5\vec{c}_5$ This represents a soln of $A\vec{x} = \vec{0}$ in which the weight of $\vec{c}_4$ is nonzero. <u>BUT</u> $\vec{v}_h$ gives <u>ALL</u> soln's of $A\vec{x} = \vec{0}$, and in $\vec{v}_h$, x_4 is ALWAYS 0, never -1, a contradiction. So $\vec{c}_4$ is <u>NOT</u> a L.C. of the other columns.

ALTERNATE but TRICKY Using this you CAN argue that

$$\left(\left[\vec{c}_1\ \vec{c}_2\ \vec{c}_4\ \vec{c}_5 \,\middle|\, \vec{c}_3\right]\right) \sim \begin{array}{c} \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \vec{c}_4 & \vec{c}_5 & | & \vec{c}_3 \end{bmatrix} \\ \left[\begin{array}{cccc|c} 1 & 9 & 0 & -7 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

but this is not quite RREF'd:

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 9 & 0 & 0 & 7/2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ x_1\ \ x_2\ \ x_4\ \ x_5 \end{array}$$

NOTE here only x_2 is free. take it as 0 get $x_1 = 7/2$, $x_2 = 0$, $x_4 = 0$, $x_5 = 1/2$

coln # 1c

1e. Show how to express $\mathbf{b}$ as a linear combination of all p columns in such a way that none of the weights involved are 0.

This means, find a soln of $A\vec{x} = \vec{b}$ in which <u>none</u> of the weights in $\vec{x}$ are 0.

Try $x_2 = x_5 = 1$; then $\vec{x} = \vec{p} + \vec{v}_h = \begin{bmatrix} 11 \\ 0 \\ -2 \\ 5 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -9 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 7 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ -4 \\ 5 \\ 1 \end{bmatrix}$, that is,

$\vec{b} = 9\vec{c}_1 + 1\vec{c}_2 - 4\vec{c}_3 + 5\vec{c}_4 + 1\vec{c}_5$ and none of the weights are 0.

2. Suppose $T : \mathbb{R}^a \to \mathbb{R}^z$ is a transformation. Give the definitions of each of the following:

2a. T is a *linear* transformation.

The following 2 conditions must be met:

① $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in $\mathbb{R}^a$

② $T(\alpha\vec{u}) = \alpha\, T(\vec{u})$ for all $\vec{u} \in \mathbb{R}^a$ & all scalars α.

2b. T is onto $\mathbb{R}^z$.

T is onto $\mathbb{R}^z \Leftrightarrow$ given ANY $\vec{b} \in \mathbb{R}^z$, there is at least one $\vec{x}$ in $\mathbb{R}^a$ for which $T(\vec{x}) = \vec{b}$.

2c. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2x_3 + 2x_1 \\ 0 \\ x_1 + x_2 + x_3 \\ 2x_2 + 7 \end{bmatrix}$. Show by example that T is not a linear transformation and that it actually fails both parts of the definition in (2a).

Does $T\left(\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}4\\5\\6\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) + T\left(\begin{bmatrix}4\\5\\6\end{bmatrix}\right)$? NO, as follows:

The left side is $T\left(\begin{bmatrix}5\\7\\9\end{bmatrix}\right) = \begin{bmatrix}7\cdot 9 + 2\cdot 5\\0\\5+7+9\\2\cdot 7+7\end{bmatrix} = \begin{bmatrix}63+10\\0\\21\\21\end{bmatrix} = \begin{bmatrix}73\\0\\21\\21\end{bmatrix}$

and the right side is $T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) + T\left(\begin{bmatrix}4\\5\\6\end{bmatrix}\right) = \begin{bmatrix}8\\0\\6\\11\end{bmatrix} + \begin{bmatrix}38\\0\\15\\17\end{bmatrix} = \begin{bmatrix}46\\0\\21\\28\end{bmatrix}$

and $\begin{bmatrix}73\\0\\21\\21\end{bmatrix} \neq \begin{bmatrix}46\\0\\21\\28\end{bmatrix}$

Does $T\left(10\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = 10\,T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)$? No as follows:

$T\left(10\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = T\left(\begin{bmatrix}10\\20\\30\end{bmatrix}\right) = \begin{bmatrix}620\\0\\60\\47\end{bmatrix}$ where as $10\,T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = 10\begin{bmatrix}8\\0\\6\\11\end{bmatrix} = \begin{bmatrix}80\\0\\60\\110\end{bmatrix}$

[note there are ∞-many examples one can give here!]

3. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is the linear transformation whose standard matrix is $A = \begin{bmatrix} 5 & 11 & 23 \\ 5 & 14 & 17 \\ 2 & 6 & 6 \\ 3 & 8 & 11 \end{bmatrix}$.

3a. Find $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$. $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} =$ "one copy of col 1 + one copy of col. 3" $= \begin{bmatrix} 28 \\ 22 \\ 8 \\ 14 \end{bmatrix}$

3b. Let $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ be a vector in $\mathbb{R}^4$. What, if any, conditions on $b_1, \ldots, b_4$ must be met to guarantee $\mathbf{b}$ is in the range of T? Show any matrices you use in answering this question.

We need to know conditions on $b_1, \ldots, b_4$ for which the system represented by $A\vec{x} = \vec{b}$ is consistent; a "super augmented" matrix keeps track of $b_1, \cdots, b_4$ as $A \rightsquigarrow RREF(A)$:

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & b_1 & b_2 & b_3 & b_4 \end{array}$$
$$\left[\begin{array}{ccc|cccc} 5 & 11 & 23 & 1 & 0 & 0 & 0 \\ 5 & 14 & 17 & 0 & 1 & 0 & 0 \\ 2 & 6 & 6 & 0 & 0 & 1 & 0 \\ 3 & 8 & 11 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|cccc} 1 & 0 & 9 & 0 & 0 & -4 & 3 \\ 0 & 1 & -2 & 0 & 0 & 3/2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 7/2 & -4 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right]$$

shows the system rep'd by $A\vec{x} = \vec{b}$ is consistent $\Leftrightarrow \begin{cases} 0 = b_1 + \frac{7}{2}b_3 - 4b_4 \\ 0 = b_2 - b_3 - b_4 \end{cases}$

3c. Suppose $\mathbf{d} = \begin{bmatrix} 3 \\ 12 \\ d_3 \\ d_4 \end{bmatrix}$. Use the conditions in (3b) to find all values of d_3 and d_4 for which $\mathbf{d}$ is in the range of T. (Note you will be setting up a little linear system, and you should use our linear algebra techniques to solve it).

we need $\begin{cases} 0 = 3 + \frac{7}{2}d_4 - 4d_4 \\ 0 = 12 - d_3 - d_4 \end{cases}$ The corresponding augmented matrix is:

$\left[\begin{array}{cc|c} 7/2 & -4 & -3 \\ -1 & -1 & -12 \end{array}\right]$ which has RREF $\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 6 \end{array}\right]$; i.e., $\vec{d} = \begin{bmatrix} 3 \\ 12 \\ 6 \\ 6 \end{bmatrix}$

3d. Is T onto $\mathbb{R}^4$? Explain your answer.

No; unless the conditions in 3b are met, there will be no $\vec{x}$ s.t. $T(\vec{x}) = \vec{b}$.

($A\vec{x} = \vec{b}$ will represent an inconsistent system if those conditions are not met)

3e. Is T one-to-one? Explain your answer.

No. Since x_3 is free, there are ∞-many solutions to $T(\vec{x}) = \vec{0}$.

(thus it is possible for $T(\vec{u}) = T(\vec{v})$ yet $\vec{u} \neq \vec{v}$)

4. Let $F = \begin{bmatrix} 2 & -3 & w \\ 1 & 5 & -2 \end{bmatrix}$ and $G = \begin{bmatrix} 8 & x & -6 \\ 7 & -2 & 1 \\ 4 & y & -5 \end{bmatrix}$, and $P = \begin{bmatrix} 7 & 23 & z \\ q & -12 & 9 \end{bmatrix}$; suppose $FG = P$.

Show all your work in the following:

4a. Find q. q is the "row 2, col 1" entry of P
so we multiply (row 2 of F) by (col. 1 of G) to find $q = 1\cdot 8 + 5\cdot 7 - 2\cdot 4$
$= 8 + 35 - 8$
$= 35$

4b. Find w. (row 1 of F) $*$ (col 1 of G) $= 7$ is the "best" to use, because ONLY the unknown w appears:

$2\cdot 8 - 3\cdot 7 + 4w = 7 \Rightarrow 16 - 21 + 4w = 7 \Rightarrow 4w = 7 - 16 + 21 = 12 \Rightarrow w = 3$

4c. Find z.

(row 1 of F) $*$ (col 3 of G) $= z \Rightarrow z = (2)(-6) + (-3)(1) + (w)(-5)$; since $w=3$,
$z = -12 - 3 - 15$
$= -30$

4d. Find x and y. Use linear algebra techniques to solve any system this problem requires.

(row 1 of F) $*$ (col 2 of G) $= 23$

$2x + 6 + 3y = 23$

$\boxed{2x + 3y = 17}$

(row 2 of F) $*$ (col 2 of G) $= -12$

$x - 10 - 2y = -12$

$\boxed{x - 2y = -2}$

we have 2 eqns & 2 unknowns. Soln's rep'd by $\left[\begin{array}{cc|c} 2 & 3 & 17 \\ 1 & -2 & -2 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 3 \end{array}\right]$

$\Rightarrow x = 4, \; y = 3$

5. Suppose that A is a 2×2 matrix and the following row operations convert A into I_2: First, rows 1 and 2 of A are swapped. Then 4 copies of row 1 are subtracted from row 2. Finally, row 2 is multiplied by 5.

5a. What three elementary matrices $E1$, $E2$, and $E3$ represent these three row operations, respectively?

Remember! An elementary matrix is one that is created from I by a SINGLE row operation; an elementary matrix is "one step away" from I

$\therefore$ STARTING WITH

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_1} E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$I_2 \xrightarrow{r_2 \leftarrow r_2 - 4r_1} E_2 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$$

$$I_2 \xrightarrow{r_2 \leftarrow 5r_2} E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

5b. Use the elementary matrices to find A^{-1}.

To say these three operations "turn A into I_2"

means $E_3\left(E_2\left(E_1 A\right)\right) = I_2$, so $(E_3E_2E_1)A = I_2$ $\therefore E_3E_2E_1 = A^{-1}$

And $E_3E_2E_1 = \begin{bmatrix} 0 & 1 \\ 5 & -20 \end{bmatrix}$

5c. Find A.

$$A = (A^{-1})^{-1} = \left(\begin{bmatrix} 0 & 1 \\ 5 & -20 \end{bmatrix}\right)^{-1} = \frac{1}{0(-20) - 5\cdot 1}\begin{bmatrix} -20 & -1 \\ -5 & 0 \end{bmatrix} = -\tfrac{1}{5}\begin{bmatrix} -20 & -1 \\ -5 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 1/5 \\ 1 & 0 \end{bmatrix}$$

5d. What is the determinant of A?

"$ad-bc$" [$\underline{\underline{\text{NOT}}}$ $\frac{1}{ad-bc}$!] is $4 \cdot 0 - 1 \cdot 1/5 = -1/5$

5e. Is A singular or nonsingular?

remember, our mnemonic for this was, that to have $\det(A) = 0$ was a "singular" [rare] occurance.

$\underline{\underline{\text{SO}}}$ A is $\underline{\text{non}}$ singular.