1. Suppose the solutions of a matrix equation $A\mathbf{x} = \mathbf{b}$ are written in the form $\mathbf{p} + \mathbf{v}_h$, where $\mathbf{p}$ is a particular solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}_h$ gives all solutions of the corresponding homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Suppose $\mathbf{b} = \begin{bmatrix} 2 \\ -13 \\ 0 \\ 2009 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 11 \\ 0 \\ -2 \\ 5 \\ 0 \end{bmatrix}$ and $\mathbf{v}_h = x_2 \begin{bmatrix} -9 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 7 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, where x_2 and x_5 are free.

1a. How many *rows* does A have? How many *columns*?

1b. Label the columns of A as $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p$. Is the set $S = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p\}$ *linearly independent*? Explain in terms of the definition of linear independence.

1c. Write $\mathbf{c}_3$ as a linear combination of the other columns of A, or explain why this cannot be done.

1d. Write $\mathbf{c}_4$ as a linear combination of the other columns of A, or explain why this cannot be done.

1e. Show how to express $\mathbf{b}$ as a linear combination of all p columns of A in such a way that none of the weights involved are 0.

2. Suppose $T : \mathbb{R}^a \to \mathbb{R}^z$ is a transformation. Give the definitions of each of the following:

2a. T is a *linear* transformation.

2b. T is onto $\mathbb{R}^z$.

2c. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2x_3 + 2x_1 \\ 0 \\ x_1 + x_2 + x_3 \\ 2x_2 + 7 \end{bmatrix}$. Show by example that T is not a linear transformation and that it actually fails both parts of the definition in (2a).

3. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is the linear transformation whose standard matrix is $A = \begin{bmatrix} 5 & 11 & 23 \\ 5 & 14 & 17 \\ 2 & 6 & 6 \\ 3 & 8 & 11 \end{bmatrix}$.

3a. Find $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$.

3b. Let $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ be a vector in $\mathbb{R}^4$. What, if any, conditions on $b_1, \ldots, b_4$ must be met to guarantee $\mathbf{b}$ is in the range of T? Show any matrices you use in answering this question.

3c. Suppose $\mathbf{d} = \begin{bmatrix} 3 \\ 12 \\ d_3 \\ d_4 \end{bmatrix}$. Use the conditions in (3b) to find all values of d_3 and d_4 for which $\mathbf{d}$ is in the range of T. (Note you will be setting up a little linear system, and you should use our linear algebra techniques to solve it).

3d. Is T onto $\mathbb{R}^4$? Explain your answer.

3e. Is T one-to-one? Explain your answer.

4. Let $F = \begin{bmatrix} 2 & -3 & w \\ 1 & 5 & -2 \end{bmatrix}$ and $G = \begin{bmatrix} 8 & x & -6 \\ 7 & -2 & 1 \\ 4 & y & -5 \end{bmatrix}$, and $P = \begin{bmatrix} 7 & 23 & z \\ q & -12 & 9 \end{bmatrix}$; suppose $FG = P$.

Show all your work in the following:

4a. Find q.

4b. Find w.

4c. Find z.

4d. Find x and y. Use linear algebra techniques to solve any system this problem requires.

5. Suppose that A is a 2×2 matrix and the following row operations convert A into I_2: First, rows 1 and 2 of A are swapped. Then 4 copies of row 1 are subtracted from row 2. Finally, row 2 is multiplied by 5.

5a. What three elementary matrices $E1$, $E2$, and $E3$ represent these three row operations, respectively?

5b. Use the elementary matrices to find A^{-1}.

5c. Find A.

5d. What is the determinant of A?

5e. Is A singular or nonsingular?