

Math 205A Test 1 (50 points)

Name: Solutions

- Check that you have 8 questions on three pages.
- Show all your work to receive full credit for a problem.

1. (5 points) (For this problem do all calculations by hand.) The augmented matrix of a system of equations is given below. Find all possible value(s) of h so that the system has a solution.

$$\begin{bmatrix} 1 & h & 2 \\ 2 & 5 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & h & 2 \\ 2 & 5 & 4 \end{bmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & h & 2 \\ 0 & 5-2h & 0 \end{bmatrix}$$

The system has a solution for any real value of h .

If $h = \frac{5}{2}$, then $5-2h=0$ and x_2 is a free variable.

Thus, the system has infinitely many solutions.

If $h \neq \frac{5}{2}$, then $5-2h \neq 0$. Then $(5-2h)x_2 = 0$ gives $x_2 = 0$.

We find x_1 from the first row and the system has a unique solution.

2. (5 points) Suppose A, B, C and X are invertible $n \times n$ matrices and $A(X+B)^T = CA$. Solve for X .

Multiply by A^{-1} on the left.

$$A^{-1}(A(X+B)^T) = A^{-1}CA$$

$$\text{So } (X+B)^T = A^{-1}CA \quad (\text{since } A^{-1}A = I)$$

Take transpose of both sides.

$$X+B = (A^{-1}CA)^T \quad (\text{since } ((X+B)^T)^T = X+B)$$

$$X = (A^{-1}CA)^T - B$$

$$\text{ie } X = A^T C^T (A^{-1})^T - B$$

$$\text{or } X = A^T C^T (A^T)^{-1} - B.$$

3. (6 points) Let A be a matrix such that its columns are vectors in \mathbb{R}^6 . The general solution of the equation $A\vec{x} = \vec{0}$ is as follows.

$$\begin{aligned}x_1 &= 2x_3 - x_5 \\x_2 &= x_3 + x_5 \\x_4 &= 3x_5 \\x_3, x_5 &\text{ are free}\end{aligned}$$

- (a) Are the columns of A linearly independent? Explain.

Since there are free variables in the solution of the equation $A\vec{x} = \vec{0}$, the equation has infinitely many solutions. Hence the columns of A are not linearly independent.

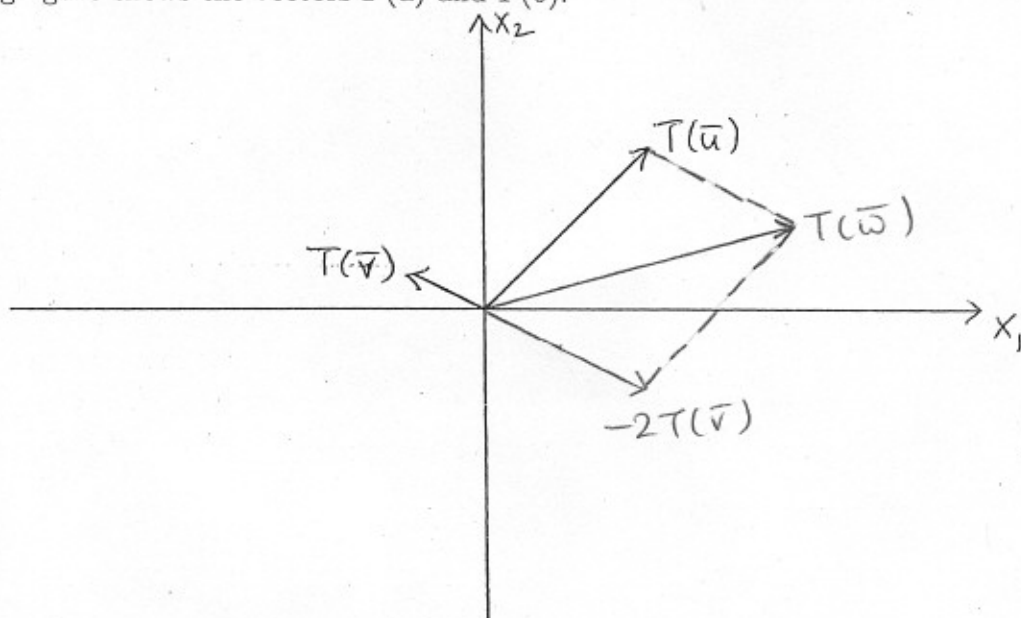
- (b) Do the columns of A span \mathbb{R}^6 ? Explain.

The reduced form of A looks as follows:

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We do not have a pivot in every row.
Hence the columns of A do not span \mathbb{R}^6 .

4. (5 points) Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation and \vec{u} and \vec{v} are two vectors in \mathbb{R}^2 . The following figure shows the vectors $T(\vec{u})$ and $T(\vec{v})$.



- (a) If $\vec{w} = \vec{u} - 2\vec{v}$, draw $T(\vec{w})$ in the above figure. Show clearly how you draw $T(\vec{w})$ and state clearly what you know about $T(\vec{w})$.

$$T(\vec{w}) = T(\vec{u} - 2\vec{v}) = T(\vec{u}) - 2T(\vec{v})$$

- (b) Is $\{T(\vec{u}), T(\vec{v})\}$ a linearly independent set? Explain.

From the figure we see that $T(\vec{u})$ is not a multiple of $T(\vec{v})$.

Hence the set $\{T(\vec{u}), T(\vec{v})\}$ is ~~not~~ a linearly independent set.

5. (10 points) Let $A = \begin{bmatrix} -6 & -3 \\ 11 & 5.5 \\ 2 & 1 \end{bmatrix}$. Let T be a linear transformation given by $T(\vec{x}) = A\vec{x}$

(a) Describe the vectors in $\text{Nul } A$ in parametric vector form.

$\text{Nul } A$ is the set of solutions to the equation $A\vec{x} = \vec{0}$.

$$\begin{bmatrix} -6 & -3 & 0 \\ 11 & 5.5 & 0 \\ 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -0.5x_2$$

x_2 is free.

Parametric vector form: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -0.5x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$

(b) Is T one-to-one? Explain.

From part (a), we see that the equation $T(\vec{x}) = \vec{0}$ i.e. $A\vec{x} = \vec{0}$ has infinitely many solutions.

Hence T is not one-to-one.

(c) Find three distinct non-zero vectors in $\text{Col } A$.

$$\text{Col } A = \text{Span} \left\{ \begin{bmatrix} -6 \\ 11 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 5.5 \\ 1 \end{bmatrix} \right\} \quad \text{Let } \vec{a}_1 = \begin{bmatrix} -6 \\ 11 \\ 2 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} -3 \\ 5.5 \\ 1 \end{bmatrix}$$

Thus every linear combination of the two column vectors is in $\text{Col } A$.

Three distinct non-zero vectors in $\text{Col } A$:

$$1 \cdot \vec{a}_1 + 0 \cdot \vec{a}_2 = \begin{bmatrix} -6 \\ 11 \\ 2 \end{bmatrix}, \quad 0 \cdot \vec{a}_1 + 1 \cdot \vec{a}_2 = \begin{bmatrix} -3 \\ 5.5 \\ 1 \end{bmatrix}$$

$$1 \cdot \vec{a}_1 + 1 \cdot \vec{a}_2 = \begin{bmatrix} -9 \\ 16.5 \\ 3 \end{bmatrix}$$

6. (5 points) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a transformation such that

$$T(x_1, x_2, x_3) = (9x_3 - x_1, 4x_2x_3 + x_1).$$

Is T a linear transformation? Explain.

For T to be a linear transformation, $T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$ for every \bar{u}, \bar{v} in \mathbb{R}^3 .

$$\text{Let } \bar{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \bar{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}. \text{ Then } T(\bar{u}) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}, T(\bar{v}) = \begin{bmatrix} 17 \\ 9 \end{bmatrix}$$

$$\text{So } T(\bar{u}) + T(\bar{v}) = \begin{bmatrix} 25 \\ 14 \end{bmatrix}.$$

$$\bar{u} + \bar{v} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}. T(\bar{u} + \bar{v}) = \begin{bmatrix} 25 \\ 26 \end{bmatrix} \neq \begin{bmatrix} 25 \\ 14 \end{bmatrix}.$$

Thus $T(\bar{u} + \bar{v}) \neq T(\bar{u}) + T(\bar{v})$ for this particular \bar{u} and \bar{v} . Hence T is not a linear transformation.

7. (5 points) Let $H = \left\{ \begin{bmatrix} b+d \\ a-d \\ a-2c+d \\ c+3d \end{bmatrix} : a, b, c, d \text{ are real numbers.} \right\}$.

Is H a subspace of \mathbb{R}^4 ? Explain.

$$\begin{bmatrix} b+d \\ a-d \\ a-2c+d \\ c+3d \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

Thus every vector in H is a linear combination of the vectors $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$.

So $H = \text{Span} \{ \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4 \}$, and hence H is a subspace of \mathbb{R}^4 .

8. (9 points) A mining company has three mines. One day of operation at the mines produces the following output.

Mine 1 produces 20 tons of copper, 600 kilograms of silver and 10 tons of manganese.

Mine 2 produces 30 tons of copper, 500 kilograms of silver and 14 tons of manganese.

Mine 3 produces 25 tons of copper, 550 kilograms of silver and 12 tons of manganese.

Suppose the company has orders for 545 tons of copper, 11350 kilograms of silver and 260 tons of manganese.

- (a) Write a vector equation to answer the question: how many days should the company operate each mine to exactly fill the orders? State clearly what the variables represent.

Let x_1 be the number of days of operation of mine 1,
 x_2 be the number of days of operation of mine 2 and
 x_3 be the number of days of operation of mine 3.
Then the vector equation is

$$x_1 \begin{bmatrix} 20 \\ 600 \\ 10 \end{bmatrix} + x_2 \begin{bmatrix} 30 \\ 500 \\ 14 \end{bmatrix} + x_3 \begin{bmatrix} 25 \\ 550 \\ 12 \end{bmatrix} = \begin{bmatrix} 545 \\ 11350 \\ 260 \end{bmatrix}$$

- (b) Give two possible answers to the question in part (a). In how many ways can we answer the question in part(a)? Explain.

$$\begin{bmatrix} 20 & 30 & 25 & 545 \\ 600 & 500 & 550 & 11350 \\ 10 & 14 & 12 & 260 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 & 17/2 \\ 0 & 1 & 1/2 & 25/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \frac{17}{2} - \frac{1}{2}x_3$$

$$x_2 = \frac{25}{2} - \frac{1}{2}x_3$$

x_3 free.

Two possible answers:

$$x_3 = 1 \text{ gives } x_1 = 8, x_2 = 12.$$

$$x_3 = 3 \text{ gives } x_1 = 7, x_2 = 11.$$

We can answer the question in many ways, as long as we choose x_3 to be between 0 and 17 (so that x_1 and x_2 are positive.)