

Math 205 (Winter 2016)

Test 1 (50 points)

Name: Solutions

- Check that you have 7 questions on three pages.
- Show all your work to receive full credit for a problem. Points will be taken off if you do not show how you arrived at your answer, even if the answer is correct.
- Please keep your explanations brief; be clear and to the point. Points will be taken off for incorrect or irrelevant statements.

1. (6 points) Consider the following system of equations.

$$x_1 + kx_2 = 1$$

$$3x_1 + 5x_2 = 2k$$

- (a) Find the row echelon form (REF) of the augmented matrix of the system given in the problem. Do this by hand and not with a calculator. State the row operation(s) that you do in your computation.

$$\begin{bmatrix} 1 & k & 1 \\ 3 & 5 & 2k \end{bmatrix} \xrightarrow{R_2 = R_2 - 3R_1} \begin{bmatrix} 1 & k & 1 \\ 0 & 5-3k & 2k-3 \end{bmatrix}$$

- (b) Find all real values of k such that the system given in the problem has only one solution.

For the system to have only one solution, the system should be consistent and should have no free variables. So there should be a pivot in the second row, second column position.

$$\text{So } 5-3k \neq 0 \text{ gives } k \neq \frac{5}{3}.$$

Thus all for the system to have only one solution, k can be any real number except $\frac{5}{3}$.

2. (8 points) A mining company has three mines. One day of operation at the mines produces the following output.

Mine 1 produces 25 tons of copper, 600 kilograms of silver and 15 tons of manganese.

Mine 2 produces 30 tons of copper, 500 kilograms of silver and 10 tons of manganese.

Mine 3 produces 20 tons of copper, 550 kilograms of silver and 12 tons of manganese.

Suppose the company has orders for 550 tons of copper, 11350 kilograms of silver and 250 tons of manganese.

(a) Write a system of equations to answer the question: how many days should the company operate each mine to exactly fill the orders? State clearly what the variables in your system represent.

Let x_1, x_2, x_3 be the number of days of operation of mine 1, 2, 3 respectively.

$$25x_1 + 30x_2 + 20x_3 = 550 \text{ order for copper.}$$

$$600x_1 + 500x_2 + 550x_3 = 11350 \text{ order for silver}$$

$$15x_1 + 10x_2 + 12x_3 = 250. \text{ order for manganese.}$$

(b) Find the general solution of the system you wrote in part (a).

$$\begin{bmatrix} 25 & 30 & 20 & 550 \\ 600 & 500 & 550 & 11350 \\ 15 & 10 & 12 & 250 \end{bmatrix} \xrightarrow[\text{with calculator}]{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

So general solution is $x_1=6, x_2=10, x_3=5$.

3. (8 points) Let $B = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 3 & 0 & 0 \\ 0 & 5 & 1 & 2 \\ 0 & -1 & 3 & 6 \end{bmatrix}$.

(a) Is B invertible? Explain.

$B \xrightarrow[\text{with calculator}]{\text{RREF}}$ $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Since B does not reduce to the identity matrix,
 B is not invertible.

(b) Define a linear transformation T by the formula $T(\vec{x}) = B\vec{x}$. Is T onto? Explain.

As seen from ^{the RREF in} part (a), B does not have a pivot position in each row.
So there exists \vec{b} in \mathbb{R}^4 such that $T(\vec{x}) = \vec{b}$ is not consistent and hence T is not onto.

4. (8 points) The solution of the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{b}$ in parametric vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 5 \\ 1 \end{bmatrix}.$$

- (a) Is the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ linearly independent? Explain.

Since the equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{b}$ has x_2 and x_4 as free variables, it has infinitely many solutions. Since the solution set of $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$ is "parallel" to the solution set of $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{b}$, the equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$ also has infinitely many solutions. Hence the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is not linearly independent.

- (b) Is \vec{v}_1 in $\text{Span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{b}\}$? Explain.

Let $x_2 = x_4 = 0$ in the parametric vector form of the soln. ~~gives~~ to $x_1\vec{v}_1 + \dots + x_4\vec{v}_4 = \vec{b}$.

Then $x_1 = 12, x_2 = 0, x_3 = 8, x_4 = 0$ is a solution to the eqn. $x_1\vec{v}_1 + \dots + x_4\vec{v}_4 = \vec{b}$.

$$\text{Hence } 12\vec{v}_1 + 8\vec{v}_3 = \vec{b}$$

$$\text{So } \vec{v}_1 = \frac{1}{12}\vec{b} - \frac{8}{12}\vec{v}_3.$$

Thus \vec{v}_1 is a linear combination of \vec{v}_3 and \vec{b} and hence it is in $\text{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{b}\}$.

5. (6 points) Let $W = \left\{ \begin{bmatrix} r-s+t \\ 11s-t \\ 3r+5s \\ 7t \end{bmatrix} \text{ where } r, s, t \text{ are real numbers} \right\}$. Is W a subspace of \mathbb{R}^4 ?

Explain.

$$\begin{bmatrix} r-s+t \\ 11s-t \\ 3r+5s \\ 7t \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 11 \\ 5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 7 \end{bmatrix}$$

$\bar{v}_1 \qquad \qquad \bar{v}_2 \qquad \qquad \bar{v}_3$

Thus every vector in W is a linear combination of $\bar{v}_1, \bar{v}_2, \bar{v}_3$.

So $W = \text{Span} \{ \bar{v}_1, \bar{v}_2, \bar{v}_3 \}$ and hence it is a subspace of \mathbb{R}^4 .

6. (5 points) Suppose the third column of a matrix B is a linear combination of the first two columns of B . Let A be a matrix such that the product AB is defined. Show that the third column of AB is a linear combination of the first two columns of AB .

Let $\bar{b}_1, \bar{b}_2, \bar{b}_3$ be the first, second and third columns of B respectively.

Then $\bar{b}_3 = c_1 \bar{b}_1 + c_2 \bar{b}_2$ for some real numbers c_1, c_2 .

$$\text{So } \underbrace{A\bar{b}_3}_{\text{third column of } AB} = A(c_1 \bar{b}_1 + c_2 \bar{b}_2) = c_1 \underbrace{(A\bar{b}_1)}_{\text{first column of } AB} + c_2 \underbrace{(A\bar{b}_2)}_{\text{second column of } AB}$$

Thus the third column of AB is a linear combination of the first two columns of AB .

7. (9 points) Suppose T is a linear transformation given by the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 - 2.5x_2 + 10x_3 \\ -x_1 + 0.5x_2 - 2x_3 \end{bmatrix}.$$

(a) Find the standard matrix for T .

$$\begin{aligned} T(\vec{x}) &= \begin{bmatrix} 5x_1 - 2.5x_2 + 10x_3 \\ -x_1 + 0.5x_2 - 2x_3 \end{bmatrix} = x_1 \begin{bmatrix} 5 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2.5 \\ 0.5 \end{bmatrix} + x_3 \begin{bmatrix} 10 \\ -2 \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 5 & -2.5 & 10 \\ -1 & 0.5 & -2 \end{bmatrix}}_{\text{standard matrix for } T} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

(b) Let $\vec{u} = 7\vec{e}_1 - 4\vec{e}_2$ where \vec{e}_1 and \vec{e}_2 are the first and second columns respectively of the 3×3 identity matrix. Find $T(\vec{u})$.

$$\begin{aligned} \vec{e}_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ T(\vec{u}) &= T(7\vec{e}_1 - 4\vec{e}_2) = 7T(\vec{e}_1) - 4T(\vec{e}_2) \\ &= 7 \begin{bmatrix} 5 \\ -1 \end{bmatrix} - 4 \begin{bmatrix} -2.5 \\ 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 45 \\ -9 \end{bmatrix}. \end{aligned}$$

$T(\vec{e}_1)$ = first column of standard matrix for T .
 $T(\vec{e}_2)$ = second column of standard matrix for T .

(c) Is $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ in the range of T ? Explain.

We solve the eqn. $T(\vec{x}) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

$$\begin{bmatrix} 5 & -2.5 & 10 & -1 \\ -1 & 0.5 & -2 & 3 \end{bmatrix} \xrightarrow[\text{with calculator}]{\text{RREF}} \begin{bmatrix} 1 & -0.5 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since the second row is of the form $[0 \ 0 \ 0 \ \text{nonzero number}]$,

the equation $T(\vec{x}) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is inconsistent.

So $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is not in the range of T .