

Math 205A Test 1 (50 points)

Name: Solutions

- Check that you have 7 questions on three pages.
- Show all your work to receive full credit for a problem.

1. (8 points) Let  $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3]$  be a  $3 \times 3$  matrix. Suppose  $\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$  is a solution of the equation  $A\vec{x} = \vec{0}$ .

- (a) Are the columns of  $A$  linearly independent? Explain. If the columns are not linearly independent, express one of the columns as a linear combination of the other columns.

Since  $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$  is a soln. of  $A\vec{x} = \vec{0}$ , we have a non-trivial solution of the equation  $A\vec{x} = \vec{0}$ . So the columns of  $A$  are not linearly independent.

$$\text{We have } -1\vec{a}_1 + 0\vec{a}_2 + 2\vec{a}_3 = \vec{0}$$

$$\text{Hence } \vec{a}_1 = 2\vec{a}_3.$$

- (b) Is  $A$  invertible? Explain.

Since ~~column~~  $A\vec{x} = \vec{0}$  has a non-trivial solution,  $A$  does not have a pivot in every column. Hence  $A$  cannot reduce to the identity matrix. So  $A$  is not invertible.

2. (8 points) Let  $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} -2 \\ -7 \\ 0 \\ 1 \end{bmatrix}$ , and  $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 6 \end{bmatrix}$ . Let  $W = \text{Span} \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ .

(a) Is  $\vec{v} = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 4 \end{bmatrix}$  in  $W$ ? Explain.

We solve the equation  $x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 = \vec{v}$  to see if  $\vec{v}$  is in  $W$ .

$$[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}] \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

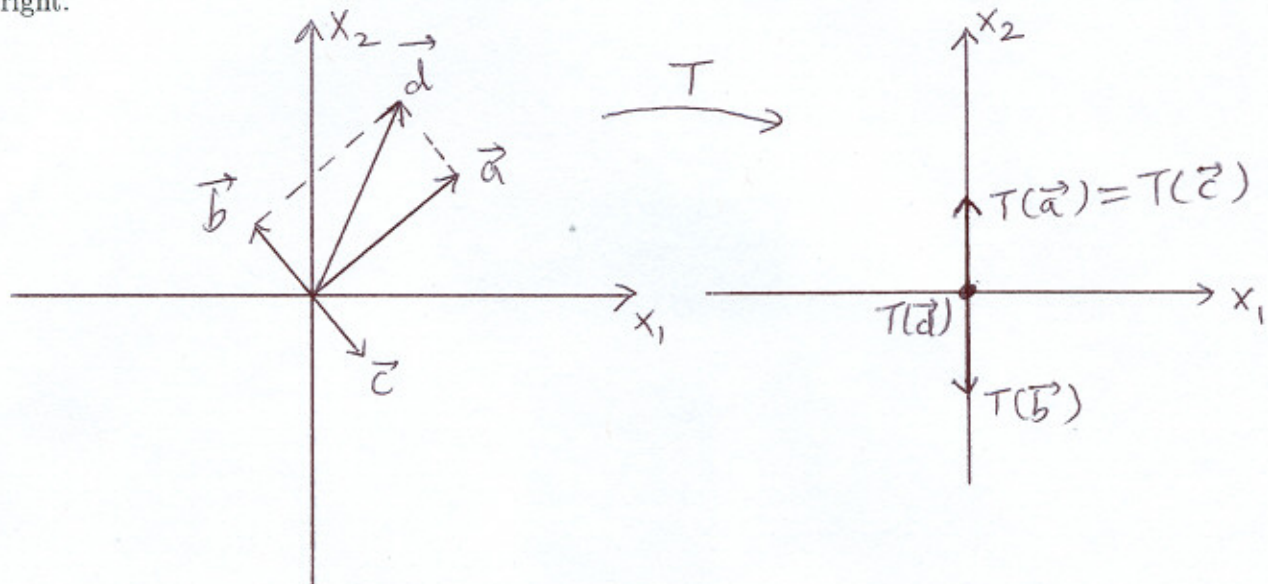
The system is consistent. So  $\vec{v}$  is in  $W$ .

(b) We know  $W$  is a subspace of  $\mathbb{R}^4$ . Does  $W = \mathbb{R}^4$ ? Explain.

$$[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since every row does not have a pivot, the set  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  does not span  $\mathbb{R}^4$ .  
Hence  $W \neq \mathbb{R}^4$ .

3. (8 points) Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear transformation. The vectors  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  and  $\vec{d}$  in  $\mathbb{R}^2$  are shown in the figure on the left. The vectors  $T(\vec{a})$  and  $T(\vec{b})$  are shown in the figure on the right.



- (a) Draw  $T(\vec{c})$  and  $T(\vec{d})$  in the figure on the right.

$$\vec{c} = -\vec{b} \quad \text{so} \quad T(\vec{c}) = -T(\vec{b})$$

$$\vec{d} = \vec{a} + \vec{b} \quad \text{so} \quad T(\vec{d}) = T(\vec{a}) + T(\vec{b})$$

- (b) Is  $T$  one-to-one? Explain.

Both  $\vec{a}$  and  $\vec{c}$  are mapped to the same vector under  $T$ , i.e.,  $T(\vec{a}) = T(\vec{c})$ .

Also both  $\vec{b}$  and  $\vec{d}$  are mapped to  $\vec{0}$  under  $T$ , i.e.,  $T(\vec{b}) = T(\vec{d}) = \vec{0}$ .

Thus,  $T$  is not one-to-one.

4. (8 points) Let  $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$  be a linear transformation such that

$$T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2 - x_3, 3x_3 + 6x_4, x_3 - x_1 - 2x_2).$$

(a) Find the standard matrix of  $T$ .

$$\begin{aligned} T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) &= \begin{bmatrix} x_1 + 2x_2 - x_3 \\ 3x_3 + 6x_4 \\ x_3 - x_1 - 2x_2 \end{bmatrix} \\ &= x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 3 & 6 \\ -1 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \end{aligned}$$

So standard matrix of  $T$  is  $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 3 & 6 \\ -1 & -2 & 1 & 0 \end{bmatrix}$ .

(b) Is  $T$  onto? Explain.

Call it  $A$ .

$$A \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since every row does not have a pivot, the equation  $A\vec{x} = \vec{b}$  is not consistent for every  $\vec{b}$  in  $\mathbb{R}^3$ .

Thus,  $T(\vec{x}) = \vec{b}$  is not consistent for every  $\vec{b}$  in  $\mathbb{R}^3$ .

Hence  $T$  is not onto.

5. (5 points) Suppose  $A, B, C$  and  $X$  are invertible  $n \times n$  matrices and  $AX + B = CX$ . Solve for  $X$ . If you need to invert a matrix, explain why that matrix is invertible.

$$AX + B = CX \text{ gives } CX - AX = B. \text{ So } (C - A)X = B.$$

To solve for  $X$ , we need to know if  $C - A$  is invertible.

$$\text{To see this, we obtain from } (C - A)X = B,$$
$$C - A = BX^{-1}.$$

$X$  is invertible. So  $X^{-1}$  is invertible.

Since  $B$  and  $X^{-1}$  are invertible,  $BX^{-1}$  is invertible.

Hence  $C - A = BX^{-1}$  is invertible.

Going back to  $(C - A)X = B$ , we get  $X = (C - A)^{-1} \cdot B$ .

6. (5 points) Let  $H = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ where } x_1 < 0, x_2 < 0, x_3 > 0 \text{ and } x_1, x_2, x_3 \text{ are real numbers.} \right\}$ .

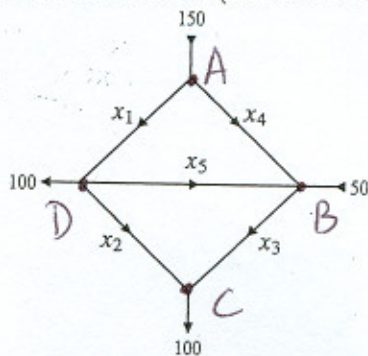
Is  $H$  a subspace of  $\mathbb{R}^3$ ? Explain.

Since  $x_1, x_2, x_3$  cannot be 0,

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  is not in  $H$ .

Hence  $H$  is not a subspace of  $\mathbb{R}^3$ .

7. (8 points) The figure below shows the traffic flow (in vehicles per half-hour) over some streets in a city.



- (a) Write a system of equations that describes the traffic flow in the above network. Remember that at each intersection the traffic flow in must equal the traffic flow out.

$$\text{At A: } 150 = x_1 + x_4$$

$$\text{At B: } x_4 + x_5 + 50 = x_3$$

$$\text{At C: } x_2 + x_3 = 100$$

$$\text{At D: } x_1 = x_2 + x_5 + 100$$

- (b) Write the general solution of the system you wrote in part (a) in parametric vector form. Based on the general solution, describe the flow of traffic (by giving values of the various variables) when the road with flow  $x_4$  is closed and the road with flow  $x_5$  has a flow of 20 vehicles per half-hour.

To find the general solution, we first rewrite the system as:

$$x_1 + x_4 = 150$$

$$x_3 - x_4 - x_5 = 50$$

$$x_2 + x_3 = 100$$

$$x_1 - x_2 - x_5 = 100$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 150 \\ 0 & 0 & 1 & -1 & -1 & 50 \\ 0 & 1 & 1 & 0 & 0 & 100 \\ 1 & -1 & 0 & 0 & -1 & 100 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 150 \\ 0 & 1 & 0 & 1 & 1 & 50 \\ 0 & 0 & 1 & -1 & -1 & 50 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{So } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_4 + 150 \\ -x_4 - x_5 + 50 \\ x_4 + x_5 + 50 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 150 \\ 50 \\ 50 \\ 0 \\ 0 \end{bmatrix}$$

When  $x_4$  is closed,  $x_4 = 0$ . So  $x_4 = 0$ ,  $x_5 = 20$ , gives

$$\vec{x} = 20 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 150 \\ 50 \\ 50 \\ 0 \\ 0 \end{bmatrix} \text{ Hence } x_1 = 150, x_2 = 30, x_3 = 70$$