Math 205 (Winter 2011)

Test 1 (50 points)

Name: Solutions.

- Check that you have 7 questions on three pages.
- Show all your work to receive full credit for a problem.

1. (8 points) Hurricanes develop low pressure at their centers that generates high winds. The maximum wind speed s (in knots) and the central pressure p of a hurricane are approximately related by the equation $a + bp = s$. We have the following data on four recent Atlantic hurricanes in the United States.

p	905	920	960	990
s	130	110	80	60

(a) Use the data to write a linear system of four equations which might be used to determine a and b.

$$a + 905b = 130$$
$$a + 920b = 110$$
$$a + 960b = 80$$
$$a + 990b = 60$$

(b) Is the system you wrote in part (a) consistent? Explain.

Augmented matrix of system in part (a) is

$$\begin{bmatrix} 1 & 905 & 130 \\ 1 & 920 & 110 \\ 1 & 960 & 80 \\ 1 & 990 & 60 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since there is a pivot in the last column, system is inconsistent.

2. (9 points) Suppose B is a 4×4 matrix with columns $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$, and $\vec{b}_4$. The solution of the equation $B\vec{x} = \vec{0}$ is given below in parametric vector form.

$$\vec{x} = x_2 \begin{bmatrix} -1.5 \\ 1 \\ 3 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0.5 \\ 0 \\ 5 \\ 1 \end{bmatrix}.$$

This should have been a 0. But it does not affect the solution.

(a) Is $\vec{b}_1$ in Span $\{\vec{b}_2, \vec{b}_3, \vec{b}_4\}$? Explain.

Let $x_2 = 1$, $x_4 = 0$.

Then $\bar{x} = \begin{bmatrix} -1.5 \\ 1 \\ 3 \\ 0 \end{bmatrix}$ is a solution of $B\bar{x} = \bar{0}$.

So $-1.5\bar{b}_1 + \bar{b}_2 + 3\bar{b}_3 = \bar{0}$

So $\bar{b}_1 = \dfrac{-\bar{b}_2 - 3\bar{b}_3}{-1.5}$. Thus, $\bar{b}_1$ is a linear combination of $\bar{b}_2, \bar{b}_3, \bar{b}_4$. So $\bar{b}_1$ is in Span $\{\bar{b}_2, \bar{b}_3, \bar{b}_4\}$.

(b) Suppose $\vec{b}$ is a vector in $\mathbb{R}^4$ such that the equation $B\vec{x} = \vec{b}$ is consistent. How many solutions does the equation have? Explain.

$B\bar{x} = \bar{0}$ has infinitely many solutions.

Solution sets of $B\bar{x} = \bar{0}$ and $B\bar{x} = \bar{b}$ are "parallel".

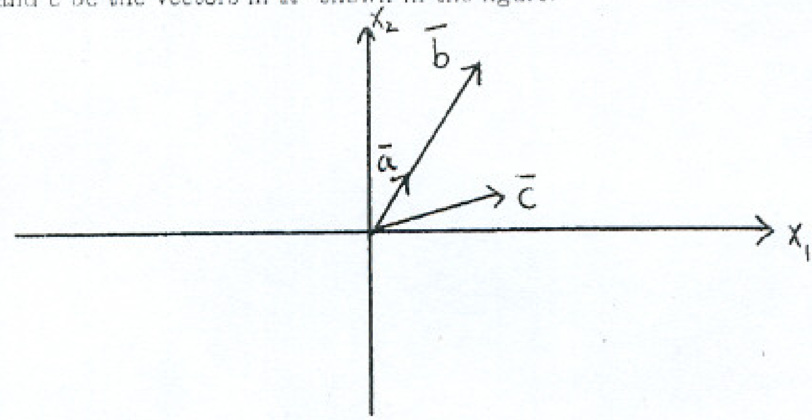
So $B\bar{x} = \bar{b}$ has infinitely many solutions.

(c) Is B invertible? Explain.

There are free variables in $B\bar{x} = \bar{0}$.

So the RREF of B does not have a pivot in every column. Thus B does not reduce to the identity matrix. Hence B is not invertible.

3. (8 points) Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be the vectors in $\mathbb{R}^2$ shown in the figure.



(a) Give a geometric description of Span$\{\vec{b}\}$.

Span $\{\bar{b}\}$ is a line in $\mathbb{R}^2$ passing through the origin and $\bar{b}$.

(b) Is the set $\{\vec{a}, \vec{c}\}$ linearly independent? Explain.

$\bar{c}$ is not a multiple of $\bar{a}$. So the eqn. $x_1\bar{a} + x_2\bar{c} = \bar{0}$ has only the trivial soln. Hence the set $\{\bar{a}, \bar{c}\}$ is lin. independent.

(c) Is the set $\{\vec{a}, \vec{b}\}$ linearly independent? Explain.

$\bar{b} = 3\bar{a}$ (this is an estimate from the figure.)

Thus, $\{\bar{a}, \bar{b}\}$ is not lin. ind.

(d) Write a non-trivial solution of the vector equation $x_1\vec{a} + x_2\vec{b} + x_3\vec{c} = \vec{0}$.

$\bar{b} = 3\bar{a}$

so $3\bar{a} - \bar{b} = \bar{0}$ i.e $3\bar{a} - \bar{b} + 0\cdot\bar{c} = \bar{0}$

$x_1 = 3, x_2 = -1, x_3 = 0$ is a non-trivial solution.

4. (5 points) Suppose T is a transformation given by the formula $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 + x_2 \\ x_2 \\ x_1 - x_2 \end{bmatrix}$.

(a) What are the domain and codomain of T?

Domain is $\mathbb{R}^2$.

Codomain is $\mathbb{R}^3$.

(b) Show that T is not a linear transformation by providing a counterexample.

One possible counterexample:

Let $\bar{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

$T(\bar{u}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $T(\bar{v}) = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$ $\bar{u} + \bar{v} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. $T(\bar{u}+\bar{v}) = \begin{bmatrix} 9 \\ 0 \\ 3 \end{bmatrix}$.

$T(\bar{u}) + T(\bar{v}) = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 9 \\ 0 \\ 3 \end{bmatrix}$. Thus, $T(\bar{u}) + T(\bar{v}) \neq T(\bar{u}+\bar{v})$.

5. (4 points) Suppose the second column of a matrix B is twice the first column and the sum of the first three columns of B is the zero vector. Let A be a matrix such that the product AB is defined.

(a) Show that the second column of AB is twice the first column of AB.

Let $\bar{b}_1, \bar{b}_2, \bar{b}_3$ be the first three columns of B.

Given: $\bar{b}_2 = 2\bar{b}_1$. So $A\bar{b}_2 = A(2\bar{b}_1) = 2(A\bar{b}_1)$.

ie Second column of $AB = 2\begin{pmatrix} \text{first column} \\ \text{of } AB \end{pmatrix}$.

(b) Show that the sum of the first three columns of AB is the zero vector.

First three columns of AB are $A\bar{b}_1, A\bar{b}_2, A\bar{b}_3$.

So sum of first three columns

$= A\bar{b}_1 + A\bar{b}_2 + A\bar{b}_3$

$= A(\bar{b}_1 + \bar{b}_2 + \bar{b}_3)$

$= A(\bar{0})$ (since $\bar{b}_1 + \bar{b}_2 + \bar{b}_3 = \bar{0}$).

$= \bar{0}$.

6. (9 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that $T(\vec{e}_1) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ and $T(\vec{e}_3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(a) Find the standard matrix of T.

Let A be the standard matrix of T.

Then columns of A are $T(\bar{e}_1), T(\bar{e}_2), T(\bar{e}_3)$.

So $A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & 1 \end{bmatrix}$

(b) Find $T(\vec{v})$ where $\vec{v} = \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix}$.

$$T(\bar{v}) = A\bar{v} = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} -3+0+7 \\ 6+0+7 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}.$$

(c) Is T onto? Explain.

$$A \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since there is a pivot in every row,

$A\bar{x} = \bar{b}$ is consistent for every $\bar{b}$ in $\mathbb{R}^2$.

So $T(\bar{x}) = \bar{b}$ is consistent for every $\bar{b}$ in $\mathbb{R}^2$.

Hence T is onto.

7. (5 points) Let $\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$, and $\vec{a}_3 = \begin{bmatrix} 2 \\ h \\ 9 \end{bmatrix}$.

(a) Let $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]$, i.e., A is the matrix with $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ as its columns. Find the row echelon form (REF) (**NOT** RREF) of A. Show all the calculations by hand.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & h \\ 0 & 5 & 9 \end{bmatrix} \xrightarrow{R3 = R3 - 5R2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & h \\ 0 & 0 & 9-5h \end{bmatrix}$$

(b) Find all possible value(s) of h such that the set $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is linearly independent.

The set is lin. ind. if the equation $A\bar{x} = \bar{0}$ has only one solution, i.e., if there are no free variables in $A\bar{x} = \bar{0}$.

From the REF in part (a), we see that there will be a pivot in ~~this~~ the third column if

$$9 - 5h \neq 0$$

i.e. if $h \neq \frac{9}{5}$.

Thus, all real values of h (except $\frac{9}{5}$) make the set lin. ind.