MATH 205 EXAM 1 FEBRUARY 9, 2007

NAME: KEY

YOUR GRADE IS BASED ON CORRECTNESS, COMPLETENESS, AND CLARITY ON EACH EXERCISE. YOU MAY USE A CALCULATOR, BUT NO NOTES, BOOKS, OR OTHER STUDENTS. GOOD LUCK!

1.) (15 pts.) Write the following system as a matrix, and row reduce (by hand) far enough to determine if the system is consistent. Do not completely solve the system.

$$\begin{array}{rcrcrcrcr} x_1 & & & & & - & 2x_4 & = & -3 \\ & & 2x_2 & + & 2x_3 & & & = & 0 \\ & & & & x_3 & + & 3x_4 & = & 1 \\ -2x_1 & + & 3x_2 & + & 2x_3 & + & x_4 & = & 5 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} \boxed{1} & 0 & 0 & -2 & -3 \\ 0 & \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

System IS consistent: there is no pivot in the augmented (last) column.

2.) (15 pts.) Let $A = \begin{bmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 2 & 1 \\ 1 & -5 & 6 \\ 7 & 3 & -3 \end{bmatrix}$.

a.) (3 pts.) Compute $A + B$. Show work for the top left entry.

$$A+B = \begin{bmatrix} -5 & -1 & -2 \\ 3 & 1 & 7 \\ 10 & 11 & 0 \end{bmatrix}$$

Top left: $(-1)+(-4) = -5$

b.) (4 pts.) Compute $AB$. Show work for the top left entry.

$$AB = \begin{bmatrix} -20 & 4 & -10 \\ 5 & -23 & 35 \\ 17 & -25 & 42 \end{bmatrix}$$

Top left:

$(-1)(-4) + (-3)(1) + (-3)(7)$

$= 4 + (-3) + (-21)$

$= -20$

c.) (4 pts.) Compute $A^{-1}$. Show how to *set up* the appropriate algorithm. You can then use a calculator to jump to the end of the algorithm.

$[A \mid I]$

$$= \left[\begin{array}{ccc|ccc} -1 & -3 & -3 & 1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 1 & 0 \\ 3 & 8 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{(Calc.)}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 3 \\ 0 & 1 & 0 & -\frac{3}{5} & \frac{6}{5} & -1 \\ 0 & 0 & 1 & -\frac{2}{5} & -\frac{1}{5} & 0 \end{array}\right] = [I \mid A^{-1}]$$

$$\underbrace{\begin{bmatrix} 2 & -3 & 3 \\ -\frac{3}{5} & \frac{6}{5} & -1 \\ -\frac{2}{5} & -\frac{1}{5} & 0 \end{bmatrix}}_{= A^{-1}}$$

d.) (4 pts.) Change the last column of $A$ so that the resulting matrix is *singular*.

Make the last column of A a linear combination of the first two columns so that A will no longer have 3 pivot positions. For example, let $\vec{a}_3 = \vec{a}_1 + \vec{a}_2$:

$$\begin{bmatrix} -1 & -3 & -4 \\ 2 & 6 & 8 \\ 3 & 8 & 11 \end{bmatrix}$$

3.) (15 pts.) Given the vectors $\mathbf{v_1} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$,

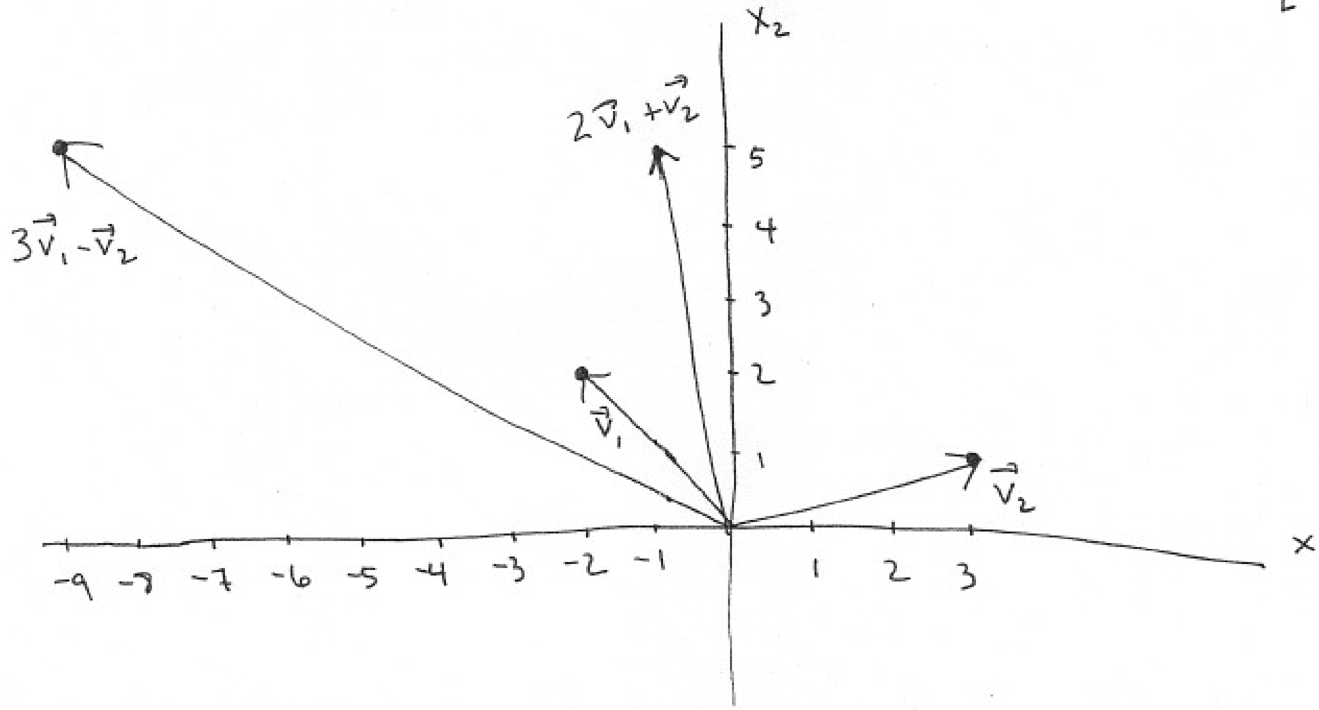
a.) (5 pts.) Sketch $\mathbf{v_1}$ and $\mathbf{v_2}$ in the same $x_1, x_2$-plane.

b.) (5 pts.) Sketch $2\mathbf{v_1} + \mathbf{v_2}$ and $3\mathbf{v_1} - \mathbf{v_2}$. (Label all sketched vectors.)

c.) (5 pts.) What is Span$\{\mathbf{v_1}, \mathbf{v_2}\}$?

$2\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

$3\vec{v}_1 - \vec{v}_2 = \begin{bmatrix} -9 \\ 5 \end{bmatrix}$




Span$\{\vec{v}_1, \vec{v}_2\}$ is $\mathbb{R}^2$ (or: is the entire $x_1, x_2$-plane)

4.) (15 pts.)

a.) (3 pts.) What is a system of equations of the form $A\mathbf{x} = \mathbf{0}$ called?

homogeneous

b.) (6 pts.) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & -1 & 3 \\ 1 & -2 & 1 & 1 \\ -2 & -1 & 1 & -3 \end{bmatrix}$. Describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

$$A \to \begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{So } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3}x_4 \\ 0 \\ \frac{1}{3}x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -4/3 \\ 0 \\ 1/3 \\ 1 \end{bmatrix}$$

c.) (6 pts.) Do the columns of $A$ span $\mathbb{R}^4$? Explain.

No. To span $\mathbb{R}^4$, there would need to be pivots in each of the four rows of A (as per Theorem 4). Since A has only three pivots, its columns do not span $\mathbb{R}^4$.

5.) (15 pts.) Let $\mathbf{u} = \begin{bmatrix} 1 \\ -4 \\ 6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 7 \\ 3 \\ -1 \end{bmatrix}$.

a.) Is $\{\mathbf{u}, \mathbf{v}\}$ linearly independent? Explain.

Yes. With just two vectors, it is enough to verify that neither is a multiple of the other. Here, it is clear that $\vec{u}$ is not a multiple of $\vec{v}$, and $\vec{v}$ is not a multiple of $\vec{u}$.

b.) Name a vector $\mathbf{w}$, not equal to either $\mathbf{u}$ or $\mathbf{v}$, that is in Span$\{\mathbf{u}, \mathbf{v}\}$. Describe how you know $\mathbf{w}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$.

Any linear combination of $\vec{u}$ and $\vec{v}$ is in Span$\{\vec{u}, \vec{v}\}$.

For example, let $\vec{w} = \vec{u} + \vec{v} = \begin{bmatrix} 8 \\ -1 \\ 5 \end{bmatrix}$.

c.) Name a vector in $\mathbb{R}^3$ that is NOT in Span$\{\mathbf{u}, \mathbf{v}\}$. Describe how you know this vector is not in Span$\{\mathbf{u}, \mathbf{v}\}$.

$\begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix}$ is not in Span$\{\vec{u}, \vec{v}\}$. There is no linear combination relation such that

$\begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix} = x_1 \vec{u} + x_2 \vec{v}$. To be certain, compute RREF of

$\begin{bmatrix} \vec{u} & \vec{v} & \begin{matrix} 8 \\ -1 \\ 4 \end{matrix} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The pivot in the last column verifies that $\begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix}$ is not a linear combination of $\vec{u}$ and $\vec{v}$.

6.) (15 pts.) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(x_1, x_2) = (5x_1 - 4x_2, -2x_1 + 3x_2, x_1 - 3x_2)$.

a.) (5 pts.) Write the standard matrix for this linear transformation.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 - 4x_2 \\ -2x_1 + 3x_2 \\ x_1 - 3x_2 \end{bmatrix} = A\vec{x} \quad \text{where} \quad A = \begin{bmatrix} 5 & -4 \\ -2 & 3 \\ 1 & -3 \end{bmatrix}.$$

A can be found as $A = [T(\vec{e}_1) \quad T(\vec{e}_2)]$.

b.) (5 pts.) Is $T(\mathbf{x})$ one-to-one? Explain.

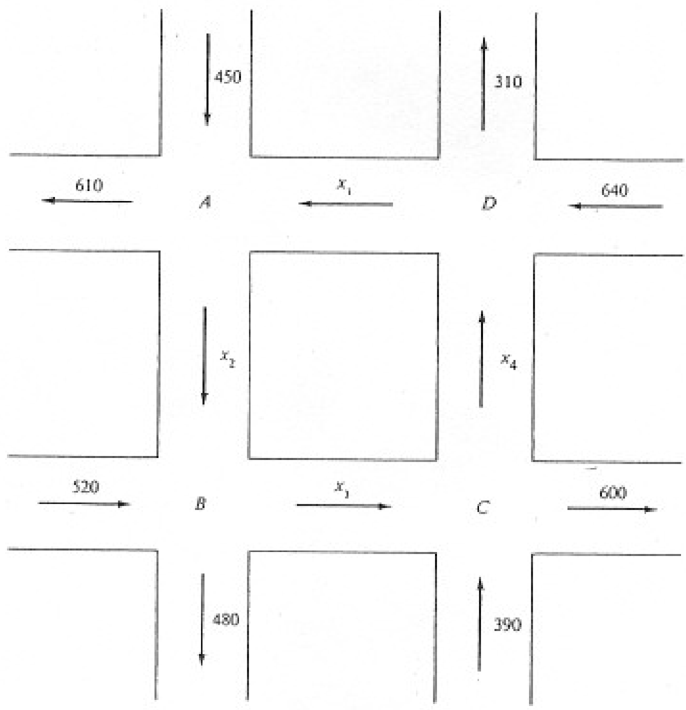
$$A \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

therefore A has no free variables, and $T(\vec{x})$ is one-to-one.

c.) (5 pts.) Is $T(\mathbf{x})$ onto? Explain.

$T(\vec{x})$ is not onto. "Onto" would mean that every possible $\vec{b}$ in $\mathbb{R}^3$ is in the range of $T(\vec{x})$, but A has only two pivots, not the three needed (one per row, as in Theorem 4).

7.) (10 pts.) In the downtown section of a certain city two sets of one-way streets intersect as shown in the figure below. The average hourly volume of traffic entering and leaving this section during rush hour is given in the diagram. Determine the amount of traffic between each of the four intersections. Then give specific amounts of traffic in the case that $x_4 = 200$.




| Intersection | Flow In | = | Flow Out | Equation, Simplified |
|---|---|---|---|---|
| A | $x_1 + 450$ | = | $x_2 + 610$ | $x_1 - x_2 = 160$ |
| B | $x_2 + 520$ | = | $x_3 + 480$ | $x_2 - x_3 = -40$ |
| C | $x_3 + 390$ | = | $x_4 + 600$ | $x_3 - x_4 = 210$ |
| D | $x_4 + 640$ | = | $x_1 + 310$ | $-x_1 + x_4 = -330$ |

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ -1 & 0 & 0 & 1 & -330 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 & 330 \\ 0 & 1 & 0 & -1 & 170 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_4 + 330$
$x_2 = x_4 + 170$
$x_3 = x_4 + 210$
$x_4$ is free

If $x_4 = 200$:
$x_1 = 530$
$x_2 = 370$
$x_3 = 410$