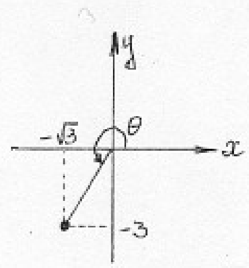
MATH 206, Section A

Test # 1

SOLUTIONS

1. (20 points) Given a rectangular point, change it to cylindrical **and** spherical coordinates.
$(-\sqrt{3},-3,-2) \Longrightarrow x=-\sqrt{3}, \quad y=-3, \quad z=-2$
a) First of all, $r=\sqrt{x^2+y^2}=\sqrt{3+9}=\sqrt{12}=2\sqrt{3}$. Since the point is in the third quadrant (both x and y are negative), we know that

$$\theta=\pi+\arctan\frac{y}{x}=\pi+\arctan\frac{-3}{-\sqrt{3}}=\pi+\arctan\sqrt{3}=\pi+\frac{\pi}{3}=\frac{4\pi}{3}.$$



Thus $\boxed{\text{cylindrical coordinates: } \left(2\sqrt{3},\frac{4\pi}{3},-2\right)}$.
b) First of all, $\rho=\sqrt{x^2+y^2+z^2}=\sqrt{3+9+4}=\sqrt{16}=4$. We know θ form (a), and

$$\varphi=\arccos\frac{z}{\rho}=\arccos\frac{-2}{4}=\arccos\left(-\frac{1}{2}\right)=\frac{2\pi}{3}.$$

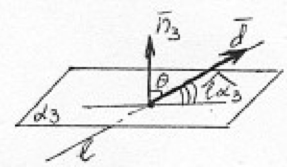
Thus $\boxed{\text{spherical coordinates: } \left(4,\frac{4\pi}{3},\frac{2\pi}{3}\right)}$.

2. (20 points) Let $\alpha_1: x-2y+z=1$, $\alpha_2: 2x+y+z=7$, and $\alpha_3: x-5y+3z=4$. Let ℓ be the line of intersection of α_1 and α_2. Find parametric equations of ℓ **and** the angle between ℓ and α_3.
a) First of all, let us find a point on ℓ : $\begin{cases} x-2y+z=1 \\ 2x+y+z=7 \end{cases}$. Since we need only a point, we can set, for example, $z=0$, then $x=1+2y$ from the first equation, and plug it into the second one, thus $2(1+2y)+y=7 \Longleftrightarrow 5y=5$, hence $y=1$ and $x=3$, therefore, $P(3,1,0)\in\ell$. A direction vector is

$$\mathbf{d}=\mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix}=\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{i}-\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}\mathbf{j}+\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}\mathbf{k}=-3\mathbf{i}+\mathbf{j}+5\mathbf{k}.$$

Thus $\boxed{\ell: x=3-3t, \quad y=1+t, \quad z=5t}$.
b) Let θ be the angle between $\mathbf{d}$ and $\mathbf{n}_3$, then

$$\cos\theta=\frac{\mathbf{d}\cdot\mathbf{n}_3}{\|\mathbf{d}\|\cdot\|\mathbf{n}_3\|}=\frac{-3-5+15}{\sqrt{9+1+25}\sqrt{1+25+9}}=\frac{7}{\left(\sqrt{35}\right)^2}=\frac{1}{5}>0,$$



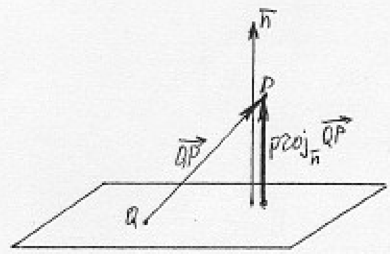
hence $\theta\in\left[0,\frac{\pi}{2}\right]$, therefore, $\widehat{\ell\alpha_3}=\frac{\pi}{2}-\theta=\boxed{\frac{\pi}{2}-\arccos\left(\frac{1}{5}\right)}$.

3. (20 points) Find a point on the curve $\mathbf{r}(t)=\left\langle 2\ln t, 4\arctan t, t^2\right\rangle$ such that the tangent line at the point is perpendicular to the plane $\langle x,y,z\rangle=\langle 1,0,3\rangle+t\langle 1,0,-1\rangle+s\langle 0,1,-1\rangle$. Find the distance between the point and the plane.
First of all, let us find a normal vector to the plane, it's the cross product of the vectors that span the plane:

$$\mathbf{n}=\langle 1,0,-1\rangle\times\langle 0,1,-1\rangle=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}=\begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix}\mathbf{i}-\begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix}\mathbf{j}+\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k}=\mathbf{i}+\mathbf{j}+\mathbf{k}.$$

Thus we want to find $t>0$ corresponding to the point P such that $\mathbf{r}'(t)$ is proportional to $\mathbf{n}$:

$$\mathbf{r}'(t)=\left\langle\frac{2}{t},\frac{4}{1+t^2},2t\right\rangle=\lambda\mathbf{n}=\lambda\mathbf{i}+\lambda\mathbf{j}+\lambda\mathbf{k}\Longrightarrow\begin{cases} 2/t=\lambda \\ 4/\left(1+t^2\right)=\lambda \\ 2t=\lambda \end{cases}\Longrightarrow\frac{2}{t}=2t\Longrightarrow t=1\Longrightarrow\boxed{P(0,\pi,1)}.$$



We know that $Q\,(1,0,3)$ is on the plane, therefore, $\overrightarrow{QP} = \langle -1, \pi, -2 \rangle$ connects the point and the plane, thus

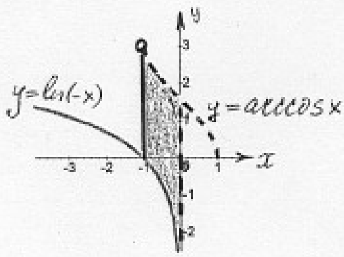
$$\text{dist} = \left\|\text{proj}_{\mathbf{n}}\overrightarrow{QP}\right\| = \left|\text{comp}_{\mathbf{n}}\overrightarrow{QP}\right| = \frac{\left|\mathbf{n}\cdot\overrightarrow{QP}\right|}{\|\mathbf{n}\|} = \frac{|-1+\pi-2|}{\sqrt{1+1+1}} = \boxed{\frac{\pi-3}{\sqrt{3}}}.$$

4. (20 points) Find the domain of the given function. Sketch the domain. Is the domain open, closed, or neither? Explain your answer. $f(x,y) = \sqrt{y - \ln(-x)} \cdot \ln(-y + \arccos x)$

$$\begin{cases} -x > 0 \\ y - \ln(-x) \geq 0 \\ x \in [-1,1] \\ -y + \arccos x > 0 \end{cases} \iff \begin{cases} x < 0 \\ y \geq \ln(-x) \\ x \in [-1,1] \\ y < \arccos x \end{cases} \implies$$

$$\boxed{\text{dom}\, f = \left\{(x,y) \in \mathbb{R}^2 : x \in [-1,0), \ln(-x) \leq y < \arccos x\right\}}.$$

The domain is neither open nor closed because there are pieces of the boundary which are included and pieces that are excluded. An open set doesn't include the boundary, while a closed set does include.



5. (20 points) For the given vector-function of three variables

$$\mathbf{F}(x,y,z) = \left\langle \ln\left(1+(xy)^2\right), \arctan\left(xz^2\right)\right\rangle$$

find its (total) derivative, the Jacobian matrix at $(-1,2,1)$, and the image of the vector $\langle 10,5,7\rangle$ under the corresponding linear transformation.

Observe that $\mathbf{F}(x,y,z) = \langle f, g\rangle$ where $f = \ln\left(1+(xy)^2\right)$ and $g = \arctan\left(xz^2\right)$, the derivative is

$$[\mathbf{DF}] = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{bmatrix} = \boxed{\begin{bmatrix} \frac{2xy^2}{1+(xy)^2} & \frac{2x^2y}{1+(xy)^2} & 0 \\ \frac{z^2}{1+(xz^2)^2} & 0 & \frac{2xz}{1+(xz^2)^2} \end{bmatrix}} \implies$$

$$\mathbf{JF}(-1,2,1) = \boxed{\begin{bmatrix} \frac{-8}{5} & \frac{4}{5} & 0 \\ \frac{1}{2} & 0 & -1 \end{bmatrix}} \implies (\mathbf{JF}(-1,2,1))\left(\begin{bmatrix} 10 \\ 5 \\ 7 \end{bmatrix}\right) = \begin{bmatrix} \frac{-8}{5} & \frac{4}{5} & 0 \\ \frac{1}{2} & 0 & -1 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \\ 7 \end{bmatrix} = \boxed{\begin{bmatrix} -12 \\ -2 \end{bmatrix}}.$$

6. (20 points) Let $F = F(u,v)$ be a function of two variables. Suppose that $u = x+y$ and $v = xy$. Find $\frac{\partial^2 F}{\partial x \partial y}$.

The partial derivative of F with respect to x is

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial F}{\partial u} + y\frac{\partial F}{\partial v}.$$

Thus the second partial mixed derivative of F once with respect to x and once with respect to y is

$$\begin{aligned} \frac{\partial^2 F}{\partial x \partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u} + y\frac{\partial F}{\partial v}\right) = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u}\right) + \frac{\partial}{\partial y}\left(y \cdot \frac{\partial F}{\partial v}\right) \\ &= \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u}\right) + 1 \cdot \frac{\partial F}{\partial v} + y \cdot \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v}\right) = \left[\frac{\partial \frac{\partial F}{\partial u}}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \frac{\partial F}{\partial u}}{\partial v}\frac{\partial v}{\partial y}\right] + \frac{\partial F}{\partial v} + y\left[\frac{\partial \frac{\partial F}{\partial v}}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \frac{\partial F}{\partial v}}{\partial v}\frac{\partial v}{\partial y}\right] \\ &= \frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial u \partial v}x + \frac{\partial F}{\partial v} + y\frac{\partial^2 F}{\partial u \partial v} + y\frac{\partial^2 F}{\partial v^2}x = \boxed{\frac{\partial F}{\partial v} + \frac{\partial^2 F}{\partial u^2} + xy\frac{\partial^2 F}{\partial v^2} + (x+y)\frac{\partial^2 F}{\partial u \partial v}}. \end{aligned}$$