Math 205A Quiz 04 page 1 February 8, 2008 NAME *suggested solutions*

1. Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ is a transformation. What are the two defining properties that T must satisfy in order to be a *linear* transformation?

① $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$

② $T(\alpha \vec{v}_1) = \alpha T(\vec{v}_1)$

for all $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$.

2. Suppose that $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}-4\\-3\\-1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\5\\8\end{bmatrix}$.

2A. Find the (standard) matrix for T.

$$\begin{bmatrix}-4 & 2\\-3 & 5\\-1 & 8\end{bmatrix}$$

2B. Find $T\left(\begin{bmatrix}2\\5\end{bmatrix}\right)$.

$$= \begin{bmatrix}-4 & 2\\-3 & 5\\-1 & 8\end{bmatrix}\begin{bmatrix}2\\5\end{bmatrix} = 2\begin{bmatrix}-4\\-3\\-1\end{bmatrix} + 5\begin{bmatrix}2\\5\\8\end{bmatrix} = \begin{bmatrix}-8\\-6\\-2\end{bmatrix} + \begin{bmatrix}10\\25\\40\end{bmatrix} = \begin{bmatrix}2\\19\\38\end{bmatrix}$$

3. Suppose $T : \mathbf{R}^3 \to \mathbf{R}^3$ has (standard) matrix $A = \begin{bmatrix}2 & 10 & 7\\1 & 5 & 6\\-2 & -10 & -15\end{bmatrix}$

3A. What, if any, conditions does a vector $\mathbf{b} \in \mathbf{R}^3$ need to satisfy in order to be in the image of T? Show any RREF matrices you use in finding your answer.

$b \in \mathbb{R}^3$ is in the image of T

$\Leftrightarrow$ there's some $\vec{x} \in \mathbb{R}^3$ s.t. $T(\vec{x}) = \vec{b}$

$\Leftrightarrow$ there's some $\vec{x} \in \mathbb{R}^3$ s.t. $A\vec{x} = \vec{b}$

note that $\left[\begin{array}{ccc|ccc}1 & 5 & 0 & 0 & 5 & 2\\0 & 0 & 1 & 0 & -2/3 & -1/3\\0 & 0 & 0 & 1 & -16/3 & -5/3\end{array}\right]$ $(\circledast)$

is the RREF of $\left[A \,\middle|\, \begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right]$ and it tells us that $T(\vec{x}) = \vec{b} \Leftrightarrow$

$0 = b_1 - \frac{16}{3}b_2 - \frac{5}{3}b_3$, where $\vec{b} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$.

3B. Show that $\mathbf{b} = \begin{bmatrix}15\\-5\\25\end{bmatrix}$ does indeed satisfy the answer to 3A.

does $15 - \frac{16}{3}(-5) - \frac{5}{3}(25) = 0$? $\Leftrightarrow 45 + 16\cdot 5 - 5\cdot 25 = 0$

$\Leftrightarrow 45 + 80 - 125 = 0$

$\Leftrightarrow 125 - 125 = 0$ ✓

3C. Find just one $\mathbf{x} \in \mathbf{R}^3$ such that $T(\mathbf{x}) = \begin{bmatrix}15\\-5\\25\end{bmatrix}$.

$(\circledast)$ in (3A) tells us that (since (3B) tells us $\begin{bmatrix}15\\-5\\25\end{bmatrix}$ is in the image)

$$\begin{cases} x_1 = 5b_2 + 2b_3 - 5x_2\\ x_2 = \text{free}\\ x_3 = -\frac{2}{3}b_2 - \frac{1}{3}b_3\end{cases} \xrightarrow[x_2=0]{\text{if we set}} \begin{cases} x_1 = 5\cdot(-5) + 2\cdot 25\\ x_2 = 0\\ x_3 = -\frac{2}{3}(-5) - \frac{1}{3}(25)\end{cases} \Rightarrow \begin{cases} x_1 = 25\\ x_2 = 0\\ x_3 = -5\end{cases} \Rightarrow \vec{x} = \begin{bmatrix}25\\0\\-5\end{bmatrix}$$