MATH106A,B CALCULUS II - PROF. P. WONG

EXAM I - FEBRUARY 6, 2009

NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
Total	100	

1.(10 pts.)(a) Find the **exact value** (by the Fundamental Theorem of Calculus) of the definite integral

$$\int_1^e \frac{\ln(2x)}{x}\ dx.$$

Let $u = \ln(2x)$**. Then** $du = \frac{1}{2x} \cdot 2dx = \frac{dx}{x}$**. When** $x = 1, u = \ln 2$**; when** $x = e, u = \ln 2e = \ln 2 + 1$**.**

It follows that

$$\int_1^e \frac{\ln(2x)}{x}\ dx = \int_{\ln 2}^{\ln 2+1} u\ du = \frac{u^2}{2}\Big|_{\ln 2}^{\ln 2+1}$$
$$= \frac{1}{2}[(\ln 2 + 1)^2 - (\ln 2)^2] = \frac{1}{2}[2\ln 2 + 1] = \ln 2 + \frac{1}{2}.$$

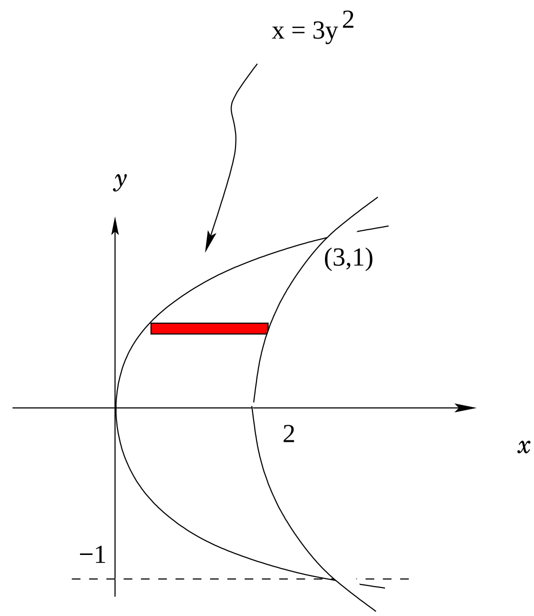
(10 pts.)(b) Evaluate the indefinite integral

$$\int xe^{(4-x^2)}\ dx.$$

Let $w = 4 - x^2$**. Thus,** $dw = -2x\ dx$ **or** $x\ dx = -\frac{1}{2}dw$**. Now,**

$$\int xe^{(4-x^2)}\ dx = -\frac{1}{2}\int e^w\ dw$$
$$= -\frac{1}{2}e^w + C$$
$$= -\frac{1}{2}e^{(4-x^2)} + C.$$

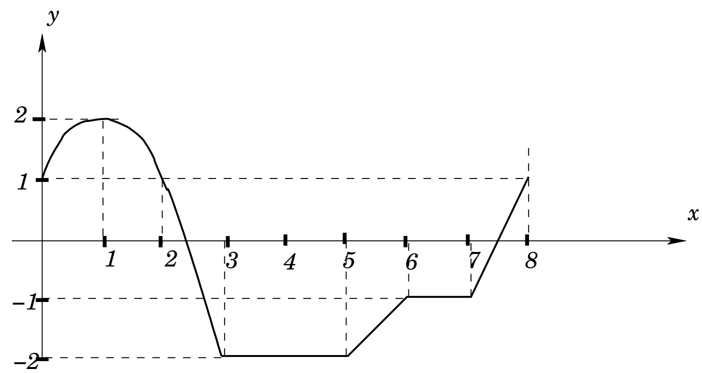
2.(20 pts.) Find the area of the region bounded by the curve $x = 3y^2$ and the curve $x = y^2 + 2$. [Hint: sketch a picture of the region by determining the points of intersection between the curves.]



When the two curves intersect, we have $3y^2 = x = y^2 + 2$ so that $2y^2 = 2$ or $y = \pm 1$ and $x = 3$. Thus, the area bounded by these curves is given by

$$\begin{aligned} A &= \int_{-1}^{1} (y^2 + 2) - 3y^2 \; dy \\ &= \int_{-1}^{1} 2 - 2y^2 \; dy = 2y - \frac{2y^3}{3} \Big|_{-1}^{1} \\ &= \left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right) = \frac{8}{3}. \end{aligned}$$

3. (10 pts.)(a) Consider a function h whose graph is given below.



Find R_8, M_4 using the right-hand sum and the mid-point rule respectively for estimating the definite integral $\int_0^8 h(x)\, dx$.

For R_8, $n = 8$ and $\Delta x = 1$ so that

$$R_8 = [2 + 1 + (-2) + (-2) + (-2) + (-1) + (-1) + 1] \cdot \Delta x = -4.$$

For M_4, $n = 4$ and $\Delta x = 2$ so that

$$M_4 = [2 + (-2) + (-2) + (-1)] \cdot \Delta x = -6.$$

(10 pts.)(b) Recall that the error committed by using the left hand sum approximation L_n is less than or equal to $\frac{K_1 \cdot (b-a)^2}{2n}$ where $|f'(x)| \le K_1$ for some constant K_1 over the interval $[a, b]$. Use this result to give an upper bound for the error committed by L_{10} for

$$I = \int_0^2 (\cos x)e^x \, dx.$$

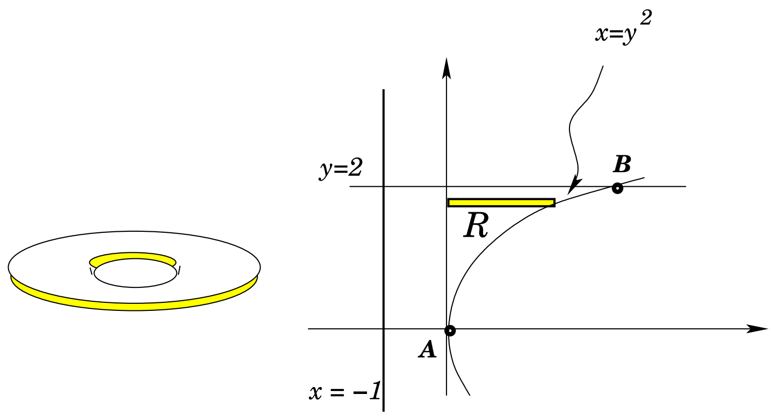
Note that $f(x) = \cos x e^x$. It follows that $f'(x) = \cos x e^x - \sin x e^x = (\cos x - \sin x)e^x$. Thus,

$$|f'(x)| \le |\cos x - \sin x|e^x \le |\cos x| + |\sin x|e^x \le (1 + 1)e^x = 2e^x.$$

Hence, over the interval $[0, 2]$, $|f'(x)| \le 2e^2$ so that we can choose $K_1 = 2e^2$. The error committed by L_{10} must be less than or equal to $\frac{2e^2(2-0)^2}{20} = \frac{2}{5}e^2$.

4. Let R be the region bounded by the curve $x = y^2$, the line $y = 2$, and the y-axis.

(15 pts.) Find the **exact volume** of the solid obtained from rotating the region R around the line $x = -1$.



Using horizontal slices each of which looks like a washer, the volume of the solid of revolution is given by

$$V = \int_0^2 \pi(y^2+1)^2 - \pi(1)^2 \ dy$$
$$= \pi \int_0^2 y^4 + 2y^2 \ dy$$
$$= \pi \left(\frac{y^5}{5} + \frac{2y^3}{3} \right)\Big|_0^2 = \frac{176\pi}{15}.$$

(5 pts.) SET UP (**do not evaluate**) a definite integral representing the arc length of the portion of the curve $x = y^2$ from A to B [Hint: What is $f(x)$ here?].

The point A **has coordinates** $(0,0)$ **and** B **has coordinates** $(4,2)$**. Here, the portion of the curve in question is the graph of** $f(x) = y = \sqrt{x}$ **from** $x = 0$ **to** $x = 4$**. Since** $f'(x) = \frac{1}{2\sqrt{x}}$**, the length of this path is given by**

$$L = \int_0^4 \sqrt{1 + [f'(x)]^2} \ dx$$
$$= \int_0^4 \sqrt{1 + \frac{1}{4x}} \ dx.$$

5. Consider the initial value problem

$$\frac{dy}{dx} = y + xy$$

with $y(0) = 1$.

(10 pts.)(a) Use Euler's method to estimate the value $y(2)$ (when $x = 2$) of the solution using two steps with initial point $(0, 1)$. DO THIS BY HAND and show all your steps.

Here the D.E. can be written as $\frac{dy}{dx} = f(x, y)$ **where** $f(x, y) = y + xy$**. With two steps from** $x = 0$ **to** $x = 2$**, the stepsize** $\Delta x = 1$**. Then,** $y_1 = y_0 + f(x_0, y_0) \cdot \Delta x$ **where** $(x_0, y_0) = (0, 1)$**. It follows that** $y_1 = 2$**. Similarly, we have** $y_2 = y_1 + f(x_1, y_1) \cdot \Delta x = 2 + (2 + 1 \cdot 2) \cdot 1 = 6$ **where** $(x_1, y_1) = (1, 2)$**. Thus,** $y(2) \approx 6$**.**

(10 pts.)(b) Use the technique of separation of variables to solve the Initial Value Problem.

Using the technique of separation of variables, we have

$$\frac{dy}{dx} = y(1 + x)$$

so that $\frac{1}{y}\ dy = (1 + x)\ dx$ **or**

$$\int \frac{1}{y}\ dy = \int (1 + x)\ dx$$

It follows that $\ln |y| = x + \frac{x^2}{2} + C$**. Since** $y(0) = 1$**, it follows that** $0 = \ln 1 = 0 + C$**. Hence,**

$$y = e^{x + \frac{x^2}{2}}.$$